

Letters to the Editor

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A MATHEMATICAL MODEL FOR MOUNTAIN WAVE IN A STRATIFIED ROTATING ATMOSPHERE ACROSS WESTERN GHATS

1. How does a fluid behave, when it crosses an obstacle ? It is a interesting problem of fluid dynamics. Challenges to this subject have been attempted by many researchers. Due to orographic waves, the pressure is systematically higher on the upwind slopes than the downward slopes and thus exerting a net force on the ground. This pressure force is known as pressure drag or mountain drag. It is one of the sinks in the atmospheric budget.

Mountain wave problem has been examined theoretically by a number of authors like Queney (1947 & 1948); Scorer (1949); Sawyer (1959); Eliassen and Palm (1961); Blumen (1965a & 1965b) ; Booker and Bretherton (1967); Jones (1967); Eliassen (1968); Bretherton (1969); Lilly (1972); Merkin and Kalnay-Rivas (1976); Buzzi and Tibaldi (1977); Mason and Sykes (1978); Smith (1978 & 1979); Eliassen and Thorsteinsson (1984); Olafsson and Bougeault (1997) etc. Queney (1947) using a two-dimensional, Boussinesq linearised model, showed that as the parameter Lf/U increases (L is mountain width, $f = 2\Omega \sin \phi$ is the coriolis parameter and U is the mean wind speed), the flow gradually loses its wavelike character in vertical x, z plane. Eliassen and Palm (1961) showed that for 2-D linear gravity waves, the vertical flux of horizontal momentum, due to waves, is independent of height, when the waves are steady and non-dissipative. Blumen (1965a) noted that the magnitude of the wave drag is sensible to the vertical wavelength. He also showed that the maximum value of the drag attained when the vertical wavelength is twice the maximum height of the mountain. Booker and Bretherton (1967) showed that vertical flux of horizontal momentum not conserved in a rotating system. Bretherton (1969) reviewed the theories concerning the propagation of internal gravity waves (IGW) in a horizontally uniform shear flow. Smith (1978) had determined the pressure drag on the Blue-ridge Mountain in the central Appalachians. During the first two weeks of January 1974, he observed several periods with significant wave drag with pressure differences typically of the order of 50 Nm^{-2} across the ridge. Smith (1979) has considered 2-D flow of a stratified rotating fluid over a ridge using linear theory model of Queney (1948). He

calculated the influence of earth's rotation on mountain wave drag and showed that coriolis force plays an important role. Olafsson and Bougeault (1997) considered a numerical model to investigate the form and magnitude of pressure drag created by elliptical mountains of various heights (h) and aspect ratios (R) in flows characterized by uniform upstream velocity (U) and stability (N). They showed that for lower value of the non-dimensional height Nh/U , the pressure drag reduced by the effect of rotation and on the other hand, for the large value of Nh/U , the rotation has the opposite effect and increases the drag. Mountain wave problem addressing properties of mountain waves over Indian region studied by many authors (Das (1964); Sarker (1965, 1966 & 1967); De (1973); Sarker *et al.* (1978); Kumar *et al.* (1995); Dutta *et al.* (2002) etc.). Dutta (2001) and Dutta & Naresh (2005) studied fluxes of momentum and energy generated by mountain waves over India.

The aim of this paper is to develop a mathematical model to obtain pressure drag, momentum and energy flux taking into account the rotation of earth for western ghats of India.

2. *The mathematical approach to the problem* - Considering the steady and frictionless flow of vertically unbounded Boussinesq fluid under the hydrostatic conditions, the governing equations for a two-dimensional north-south oriented Mumbai-Pune section of the western ghats are given by

$$\rho_0 U \frac{\partial u'}{\partial x} - \rho_0 f v' + \frac{\partial p'}{\partial x} = 0 \quad (2.1)$$

$$\rho_0 U \frac{\partial v'}{\partial x} - \rho_0 f u' = 0 \quad (2.2)$$

$$\rho_0 U \frac{\partial w'}{\partial x} + \frac{\partial p'}{\partial z} + \rho' g = 0 \quad (2.3)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (2.4)$$

$$U \frac{\partial \rho'}{\partial x} + \frac{d\bar{\rho}}{dz} w' = 0 \quad (2.5)$$

where u' , w' , p' and ρ' are the perturbation zonal wind, vertical wind, pressure and density respectively. The mean density (ρ_0), gravitational acceleration (g) and vertical density gradient $\frac{d\bar{\rho}}{dz}$ are taken as constant. Also, it is assumed that basic flow is normal to ridge and is constant with height. The value of U has been taken as the mean of winds at different level up to which westerly prevail. $f = 2\omega \sin \phi$ is constant Coriolis parameter, ω is the angular velocity of Earth and ϕ denotes the latitude.

The gravitational stability of basic state has been characterized by Brunt-Väisälä frequency $N = \left(-\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz} \right)^{1/2}$ and assumed to be constant with height. Near the ground, the vertical velocity must satisfy the boundary condition

$$w'(x, z = 0) = U \frac{\partial h}{\partial x} \tag{2.6}$$

where, $h(x)$ is the profile of western ghats, which disturbed the flow and is considered by Sarker *et al.*, (1978) is

$$h(x) = \frac{H}{1 + \frac{x^2}{a^2}} + b \tan^{-1} \frac{x}{a} \tag{2.7}$$

where $a = 18.0$ km, $H = .52$ km, $b = \frac{2}{\pi} \times .35$ km for Western Ghats section along Bombay-Pune and $a = 18.0$ km, $H = .52$ km, $b = .88$ km for E-W Western Ghats section along Manglore-Agumbe.

The Fourier transform of the equations (2.1) to (2.5) reduces to a single equation for $\hat{w}(k, z)$, which is the vertical velocity of a fluid parcel in Fourier form

$$\frac{d^2 \hat{w}}{dz^2} + k^2 \left(\frac{l^2 - k^2}{k^2 - k_f^2} \right) \hat{w} = 0 \tag{2.8}$$

where, $l = \frac{N}{U}$ is the Scorer's parameter and $k_f = \frac{f}{U}$.

The solution of Eqn. (2.8) is

$$\hat{w}(k, z) = Ae^{ik \left(\frac{l^2 - k^2}{k^2 - k_f^2} \right)^{1/2} z} + Be^{-ik \left(\frac{l^2 - k^2}{k^2 - k_f^2} \right)^{1/2} z} \tag{2.9}$$

For vertically propagating hydrostatic wave $k \ll l \approx 10^{-3} \text{ m}^{-1}$

Thus, Eqn. (2.9) reduces to

$$\hat{w}(k, z) = Ae^{ik \left(\frac{l}{k^2 - k_f^2} \right)^{1/2} z} + Be^{-ik \left(\frac{l}{k^2 - k_f^2} \right)^{1/2} z} \tag{2.10}$$

To allow the energy to propagate at great height and using Eqns. (2.6), (2.7) into (2.10), we get

$$\hat{w}(k, z) = ikU \left[aH - i \frac{b}{k} \right] e^{-ak} e^{ik \left(\frac{l}{k^2 - k_f^2} \right)^{1/2} z} \tag{2.11}$$

3. *Mountain drag* - Consider the horizontal force exerted from below across the chosen orography $h(x)$. Assume that perturbation is vanish at $x = \infty$ or $x = -\infty$

$$F = \int_{-\infty}^{\infty} p' dx = \int_{-\infty}^{\infty} p' \frac{d\eta'}{dx} dx = - \int_{-\infty}^{\infty} \eta' \frac{dp'}{dx} dx \tag{3.1}$$

where, $\eta'(x, z)$ is the height of the streamline above undisturbed level. As, near the ground $\eta'(x, z = 0) = h(x)$

$$\text{also, } w' = U \frac{\partial \eta'}{\partial x} \tag{3.2}$$

Using Eqn. (3.2) into (3.1), we get

$$F = - \int_{-\infty}^{\infty} \rho_0 u' w' dx - f \int_{-\infty}^{\infty} \rho_0 v' \eta' dx \tag{3.3}$$

Now, using Parseval's theorem for Fourier integral, The Mountain drag becomes

$$F = -2\pi \int_{-\infty}^{\infty} \rho_0 \hat{u} \hat{w}^* dk - 2\pi f \int_{-\infty}^{\infty} \rho_0 \hat{v} \hat{\eta}^* dk \tag{3.4}$$

where, \hat{w}^* and \hat{h}^* are complex conjugates of \hat{w} and \hat{h} respectively.

Using Fourier transform of Eqns. (2.2), (2.4), (2.11) and (3.2) to (3.4), we get

$$F = 2\pi\rho_0 NU \int_{-\infty}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 H^2 + \frac{b^2}{k^2} \right) e^{-2ak} dk \quad (3.5)$$

Expression (3.5) is the mountain drag in integral form.

As Dutta (2001) has shown that momentum flux is equal to the negative of the mountain drag. Therefore, momentum flux will be

$$F_1 = -2\pi\rho_0 NU \int_{-\infty}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 H^2 + \frac{b^2}{k^2} \right) e^{-2ak} dk \quad (3.6)$$

To get non-negative real solution, we need to integrate only over range of k , where $k^2 > k_f^2$

Therefore,

$$F = 2\pi\rho_0 NU \int_{k_f}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 H^2 + \frac{b^2}{k^2} \right) e^{-2ak} dk \quad (3.7)$$

For $f = 0 \Rightarrow k_f = 0$, Eqn. (3.7) becomes

$$\begin{aligned} F_{f=0} &= 2\pi\rho_0 NU \int_0^{\infty} \left(a^2 H^2 k + \frac{b^2}{k} \right) e^{-2ak} dk \\ &= \frac{1}{2} \pi\rho_0 NU H^2 \end{aligned} \quad (3.8)$$

Using Eqn. (3.8) into Eqn. (3.7), we get

$$F = 4F_{f=0} \int_{k_f}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 + \frac{b^2}{H^2 k^2} \right) e^{-2ak} dk \quad (3.9)$$

Now, substituting $p_f = 2ak_f = 2a \frac{f}{U}$ and $p = 2ak$ into Eqn. (3.9)

$$F = F_{f=0} R(p_f) \quad (3.10)$$

$$\text{where, } R(p_f) = \int_{p_f}^{\infty} (p^2 - p_f^2)^{\frac{1}{2}} \left(1 + \frac{4b^2}{p^2 H^2} \right) e^{-p} dp \quad (3.11)$$

In similar fashion, we may get

$$F_1 = F_{1f=0} R(p_f) \quad (3.12)$$

4. *Energy flux* - As shown by Eliassen and Palm (1961), the expression of vertical flux of wave energy is

$$E = \int_{-\infty}^{\infty} p' w' dx = 2\pi \int_{-\infty}^{\infty} \hat{p} \hat{w}^* dx \quad (4.1)$$

Using Fourier transform of Eqns. (2.1), (2.2) & (2.4) into Eqn. (4.1), we get

$$E = -2\pi i \rho_0 U \int_{-\infty}^{\infty} \frac{1}{k^3} (k^2 - k_f^2) \frac{\partial \hat{w}}{\partial z} \hat{w}^* dk \quad (4.2)$$

Now, substitute Eqn. (2.11) into Eqn. (4.2), we get

$$E = 2\pi\rho_0 NU^2 \int_{-\infty}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 H^2 + \frac{b^2}{k^2} \right) e^{-2ak} dk \quad (4.3)$$

For non-negative and real solution, Eqn. (4.3) becomes

$$E = 2\pi\rho_0 NU^2 \int_{k_f}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 H^2 + \frac{b^2}{k^2} \right) e^{-2ak} dk \quad (4.4)$$

Again, for $f = 0$, Eqn. (4.4) becomes

$$\begin{aligned} E_{f=0} &= 2\pi\rho_0 NU^2 \int_0^{\infty} \left(a^2 H^2 k + \frac{b^2}{k} \right) e^{-2ak} dk \\ &= \frac{1}{2} \pi\rho_0 NU^2 H^2 \end{aligned} \quad (4.5)$$

TABLE 1

Date	Time (UTC)	U	N	Present results for rotating atmosphere		Results for non-rotating atmosphere	
				F	E	F _{f=0}	E _{f=0}
04 Jan '59	1200	12.3	0.60	2320705	28544671	3381501	41592462
21 May '59	1200	13.7	0.61	2725914	37345022	3849731	52741315
06 Dec '60	1200	12.3	0.61	2379117	29263139	3466613	42639340
14 Dec '60	0000	06.7	0.61	1048232	7023154	1892394	12679040
26 Dec '60	0000	09.6	0.61	1702887	16347715	2686633	25791677
06 May '65	0000	12.8	0.61	2561961	32793101	3689302	47223066

Now, using Eqn. (4.5) into Eqn. (4.4), we get

$$E = 4E_{f=0} \int_{k_f}^{\infty} (k^2 - k_f^2)^{\frac{1}{2}} \left(a^2 H^2 + \frac{b^2}{k^2} \right) e^{-2ak} dk \quad (4.6)$$

Finally, substituting $p_f = 2ak_f = 2a \frac{f}{U}$ into Eqn. (4.6), we get,

$$E = E_{f=0} R(p_f) \quad (4.7)$$

Particular Cases - If we take $f = 0$, then the problem reduces to that considered by Dutta (2001). In this case, by putting $f = 0$ into the equations (3.8) and (4.4), they reduce to the equations (3.9) and (4.5), which are similar to the expressions of mountain drag and energy flux as obtained by Dutta (2001) for the relevant problem.

5. The expressions of mountain drag, momentum and energy flux obtained in integral form are given by equations (3.5), (3.6) and (4.3) respectively. Further, they are expressed in terms of $R(p_f)$, drag and flux for non-rotating atmosphere as given in equations (3.10), (3.12) and (4.7) respectively. $R(p_f)$ has been evaluated numerically for different wind speeds U as shown in Fig. 1.

We observe that as the mean wind U decreases the magnitude of mountain drag, momentum fluxes and energy flux decrease from their value at $f = 0$ respectively.

Also, for a quite low value of $U \approx 2.5 \text{ms}^{-1}$, the magnitude of drag and both the fluxes become less than one-third of its value at $f = 0$, while for $U \approx 6.0 \text{ms}^{-1}$, the

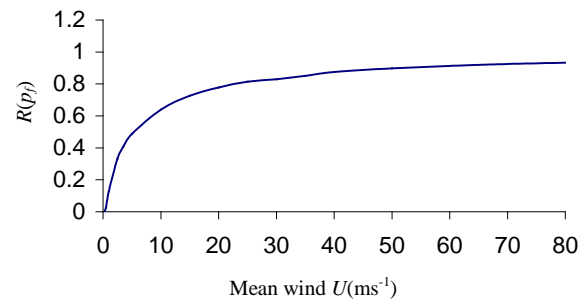


Fig. 1. Variation of $R(p_f)$ with U

magnitude of drag and both the fluxes become half of its value at $f = 0$. At very high values of U the contribution of f become very small. In turn flow becomes nearly geostrophic.

Further, we can elaborate our studies to see that as latitude increases (*i.e.*, f increases) or width of the mountain 'a' increases, the magnitude of drag and both the fluxes decrease from its value at $f = 0$ and become very small in magnitude. In turn, flow becomes nearly geostrophic. Equations (3.9) and (4.6) show that drag and both the fluxes are dependent on the half width of the bell shaped portion of western ghats in case of stratified rotating atmosphere. This is not true in case of non-rotating atmosphere as shown by equations (3.8) and (4.5). The mountain drag and energy flux have been computed by using Radiosonde data as used by Dutta (2001) and these computed drag and flux are compared with Dutta (2001) results, which is given in Table 1.

From Table 1, by comparing the results between stratified rotating atmosphere and stratified non-rotating atmosphere, it has been noticed that contribution of coriolis force f is important and cannot be ignored.

TABLE 2

Date	Time (UTC)	U	N	Results without plateau part		Results with plateau part	
				F	E	F	E
04 Jan '59	1200	12.3	0.60	3297246	40556126	2320705	28544671
21 May '59	1200	13.7	0.61	3769951	51648329	2725914	37345022
06 Dec '60	1200	12.3	0.61	3380237	41576915	2379117	29263139
14 Dec '60	0000	06.7	0.61	1765222	11826987	1048232	7023154
26 Dec '60	0000	09.6	0.61	2585289	24818774	1702887	16347715
06 May '65	0000	12.8	0.61	3603408	46123622	2561961	32793101

It is also found that the plateau part of the western ghats contributes towards the generation of the mountain drag, momentum flux and energy flux, which is absent in the case of non-rotating atmosphere as shown by Dutta (2001). Results for without plateau and with plateau for stratified atmosphere are given in Table 2.

In the Table 2 the units of U , N , F , E , $F_{f=0}$ and $E_{f=0}$ are ms^{-1} , s^{-1} , Nm^{-2} and Wm^{-2} respectively.

From Table 2, one can conclude that plateau part of the western ghats is a good contributor for generation of the mountain drag, momentum flux and energy flux. It is due to the rotation of the earth, which produces an additional drag on the plateau.

Further, equations (3.10), (3.12) and (4.7) show the following equality

$$R(p_f) = \frac{F}{F_{f=0}} = \frac{F_1}{F_{1f=0}} = \frac{E}{E_{f=0}}$$

which implies that variation of drag and both the fluxes in stratified rotating atmosphere and drag and fluxes in non-rotating atmosphere are in same ratio.

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