Some characteristic parameters of Gaussian plume model

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सार – उर्ध्वाधर सांद्रण वितरण की प्रथम चार स्थितियों (केंद्रक, भेद, वैषम्य और कर्टोसिस) को ज्ञात करने हेतु रैखिक स्रोत के लिए विसरण समीकरण के लिए गौसियन सॉल्यूशन का उपयोग किया गया है। इसमे अधिकतम सांद्रण स्तर की स्थिति एवं परिमाण की गणना की गई है। इसमें पिच्छक अभिवहन पवन गति का भी आकलन किया गया है। भूतल स्तर सांद्रण के समीकरणों की तुलना पवन टनल मापनों से की गई है।

ABSTRACT. The Gaussian solution of the diffusion equation for line source is used to have the first four moments of the vertical concentration distribution (centroid, variance, skewness, and kurtosis). The magnitude and position of maximum concentration level were evaluated. Also the plume advection wind speed is estimated. Equations for the ground level concentration were compared with wind tunnel measurements.

Key words – Moments, Diffusion equation, Plume advection wind speed.

1. Introduction

The modeling of dispersion has been performed by a Gaussian approach that takes account of atmospheric turbulence assuming simple formula for concentration distribution, in which the dispersion parameters depend on downwind distance and the Pasquill-Gifford scheme (Arya, 1999). The statistical description distribution for non Gaussian concentration model is studied by Brown *et al.* (1997). The moments and maximum ground level concentration in two dimensions are obtained by Tirabassi *et al.* (2009).

In this paper, we derive the first four moments of the vertical concentration distribution from Gaussian plume model (Pasquill and smith 1984). Equations for the maximum ground level concentration along the centerline and its locations were calculated. The effective wind speed of a plume is derived.

2. The Gaussian model

One assumes that the turbulent mass flux can be described by a gradient-transfer closure. The steady-state diffusion equation can be written as (Pasquill and Smith, 1984).

$$\overline{U}\frac{\partial}{\partial x}\overline{C} = \frac{\partial}{\partial y}\left(K_y \frac{\partial}{\partial y}\overline{C}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial}{\partial z}\overline{C}\right) + S \qquad (1)$$

where x, y, and z are the longitudinal, lateral, and vertical directions, respectively, $\overline{U}$ is the mean longitudinal wind speed, $\overline{C}$ is the mean concentration, K_y and K_z are the lateral and vertical eddy diffusivities, respectively, and S is a sink or source term. The solution of equation (1) can be written as (Pasquilll (1974) :

$$\frac{C(x,y,z)}{Q} = \frac{A}{\sqrt{2\pi}\sigma_y \overline{z}\overline{U}}\exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]\exp\left(\frac{y^2}{2\sigma_y^2}\right)$$

(2)

From this solution, one can estimate the concentration of a point source where,

Q - Actual emission rate of the point source (µg/s).

σ_y - Crosswind dispersion plume spread (m).

$\overline{U}$ - Mean wind speed (m/s).

$\overline{z}$ - Mean plume vertical height (m).

s - Parameter depends on the stability.

A, B - Parameters depend on the stability, where $B = \Gamma(2/s)/\Gamma(1/s)$ [Van Ulden (1978)].

3.   Moments of the Gaussian distribution

The moments of centroid, the variance, skewness and kurtosis can be obtained from the following equation:

$$\frac{d\overline{z}}{dx} = \frac{\int\limits_0^\infty (z-\overline{z})^m \overline{C}(x,z)dz}{\int\limits_0^\infty \overline{C}(x,z)dz} \qquad (3)$$

To measure the centroid $\overline{z}$ of the concentration distribution, put $m = 1$ and substituting from equation (2) in equation (3), we obtain that:

$$\frac{d\overline{z}}{dx} = \frac{\int\limits_0^\infty z\exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz - \int\limits_0^\infty \overline{z}\exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz}{\int\limits_0^\infty \exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz} \qquad (4)$$

Using separation of variables and integrating equation (4), we obtain the centroid as follows:

Hence,

$$\int\limits_{z_0}^{\overline{z}} \frac{d\overline{z}}{\overline{z}} = \left[\frac{\Gamma\left(\frac{2}{s}\right) - B\Gamma\left(\frac{1}{s}\right)}{B\Gamma\left(\frac{1}{s}\right)}\right]\int\limits_0^x dx$$

$$\Rightarrow In(\overline{z}) = \left[\frac{\Gamma\left(\frac{2}{s}\right) - B\Gamma\left(\frac{1}{s}\right)}{B\Gamma\left(\frac{1}{s}\right)}\right]x + In(z_0)$$

$$(5)$$

hence

$$\overline{z}(x) = \exp\left\{\left[\frac{\Gamma\left(\frac{2}{s}\right) - B\Gamma\left(\frac{1}{s}\right)}{B\Gamma\left(\frac{1}{s}\right)}\right]x + In(z_0)\right\}$$

Where : $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$ is Γ - function.

To measure the variance $\sigma_z^2(x)$ of the concentration distribution, put $m = 2$, in equation (3) and substituting from equation (2) in equation (3), we obtain that:

$$\frac{d\overline{z}}{dx} = \frac{\begin{array}{c}\int\limits_0^\infty z^2\exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz - 2\overline{z}\int\limits_0^\infty z\exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz \\ +\overline{z}^2\int\limits_0^\infty \exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz\end{array}}{\int\limits_0^\infty \exp\left[-\left(\frac{Bz}{\overline{z}}\right)^s\right]dz}$$

$$(6)$$

Using separation of variables and integrating equation (6), we obtain the variance as follows:

$$\int\limits_{z_0}^{\overline{z}} \frac{d\overline{z}}{\overline{z}^2} = \left[\frac{\Gamma\left(\frac{3}{s}\right) - 2B\Gamma\left(\frac{2}{s}\right) + B^2\Gamma\left(\frac{1}{s}\right)}{B^2\Gamma\left(\frac{1}{s}\right)}\right]\int\limits_0^x dx$$

$$\Rightarrow \frac{1}{\overline{z}} = \frac{1}{z_0} - \left\{\left[\frac{\Gamma\left(\frac{3}{s}\right) - 2B\Gamma\left(\frac{2}{s}\right) + B^2\Gamma\left(\frac{1}{s}\right)}{B^2\Gamma\left(\frac{1}{s}\right)}\right]\right\}x$$

$$(7)$$

$$\sigma_z^2(x) = \frac{z_0 B^2\Gamma\left(\frac{1}{s}\right)}{B^2\Gamma\left(\frac{1}{s}\right) - z_0 x\left[\begin{array}{c}\Gamma\left(\frac{3}{s}\right) - 2BT\left(\frac{2}{s}\right) \\ +B^2\Gamma\left(\frac{1}{s}\right)\end{array}\right]} = \overline{z}_{m=2}$$

To measure the skewness sk (*x*) of the concentration distribution, put *m*=3 in equation (3) and substituting from equation (2) in equation (3), to have:

$$\frac{d\bar{z}}{dx} = \frac{\int_0^\infty z^3 \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz - 3\bar{z}\int_0^\infty z^2 \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz + 3\bar{z}^{-2}\int_0^\infty z \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz - \bar{z}^{-3}\int_0^\infty \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz}{\int_0^\infty \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz}$$

(8)

Using separation of variables and integrating equation (8), we obtain the skewness as follows:

$$\int_{z_0}^{\bar{z}} \frac{d\bar{z}}{\bar{z}^{-3}} = \left[\frac{\Gamma\left(\frac{4}{s}\right) - 3B\Gamma\left(\frac{3}{s}\right) + 3B^2\Gamma\left(\frac{2}{s}\right) - B^2\Gamma\left(\frac{1}{s}\right)}{B^3\Gamma\left(\frac{1}{s}\right)}\right]\int_0^x dx$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{z_0^2} - \frac{1}{\bar{z}^{-2}}\right)\left[\frac{\Gamma\left(\frac{4}{s}\right) - 3B\Gamma\left(\frac{3}{s}\right) + 3B^2\Gamma\left(\frac{2}{s}\right) - B^2\Gamma\left(\frac{1}{s}\right)}{B^3\Gamma\left(\frac{1}{s}\right)}\right]x$$

(9)

$$: sk(x) = \left\{\frac{z_0^2 B^3\Gamma\left(\frac{1}{s}\right)}{B^3\Gamma\left(\frac{1}{s}\right) - \begin{bmatrix}2\Gamma\left(\frac{4}{s}\right) - 6B\Gamma\left(\frac{3}{s}\right) \\ + 6B^2\Gamma\left(\frac{2}{s}\right) \\ - 2B^3\Gamma\left(\frac{1}{s}\right)\end{bmatrix}z_0^2 x}\right\}^{\frac{1}{2}} = \bar{z}_{m=3}$$

To measure the kurtosis "ku" of the concentration distribution, put *m*=4 in equation (3) and substituting from equation (2) in equation (3), we get:

$$\frac{d\bar{z}}{dx} = \frac{\int_0^\infty z^4 \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz - 4\bar{z}\int_0^\infty z^3 \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz + 4\bar{z}^{-2}\int_0^\infty z^2 \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz + 2\bar{z}^{-3}\int_0^\infty z \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz - \bar{z}^{-4}\int_0^\infty \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz}{\int_0^\infty \exp\left[-\left(\frac{Bz}{\bar{z}}\right)^s\right]dz}$$

(10)

Separating of variables and integrating eqn.(10), we obtain the kurtosis as follow :

$$\int_{z_0}^{\bar{z}} \frac{d\bar{z}}{\bar{z}^{-4}} = \left[\frac{\begin{matrix}\Gamma\left(\frac{5}{s}\right) - 4B\Gamma\left(\frac{4}{s}\right) + 4B^2\Gamma\left(\frac{3}{s}\right) \\ + 2B^3\Gamma\left(\frac{2}{s}\right) - B^4\Gamma\left(\frac{1}{s}\right)\end{matrix}}{B^4\Gamma\left(\frac{1}{s}\right)}\right]\int_0^x dx$$

$$\Rightarrow \frac{1}{4}\left[In(\bar{z}) - z_0\right] = \left[\frac{\begin{matrix}\Gamma\left(\frac{5}{s}\right) - 4B\Gamma\left(\frac{4}{s}\right) + 4B^2\Gamma\left(\frac{3}{s}\right) \\ + 2B^3\Gamma\left(\frac{2}{s}\right) - B^4\Gamma\left(\frac{1}{s}\right)\end{matrix}}{B^4\Gamma\left(\frac{1}{s}\right)}\right]x$$

(11)

$$ku = \exp\left\{\left[\frac{\begin{matrix}4\Gamma\left(\frac{5}{s}\right) - 16B\Gamma\left(\frac{4}{s}\right) + 16B^2\Gamma\left(\frac{3}{s}\right) \\ + 8B^3\Gamma\left(\frac{2}{s}\right) - 4B^4\Gamma\left(\frac{1}{s}\right)\end{matrix}}{B^4\Gamma\left(\frac{1}{s}\right)}\right]x + z_0\right\} = \bar{z}_{m=4}$$
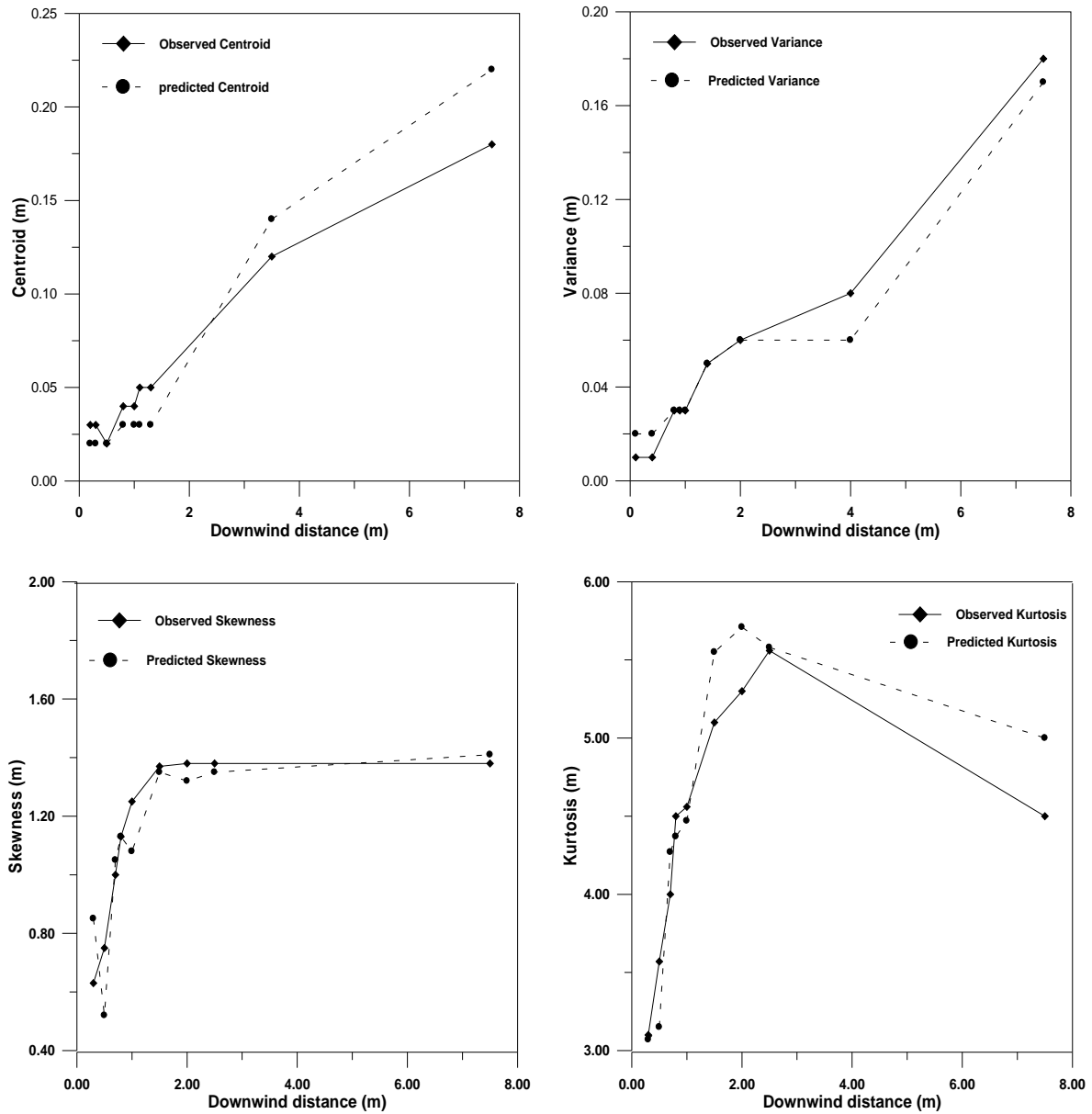
Fig. 1. Predicted centroid, variance, skewness and kurtosis of the vertical concentration distribution for near surface are compared with wind tunnel measured *via* downwind distance (Khurshudyan *et al.* 1981)

4. Maximum ground level concentration

Substituting with $z = 0$ and $\sigma_y = \dfrac{\sigma_v x}{\overline{U}}$ in equation (2), to have:

$$C_{GLC} = \frac{AQ}{\sqrt{2\pi}\,\sigma_v xz} \exp\left(-\frac{y^2 \overline{U}^2}{2\sigma_v^2 x^2}\right) \qquad (12)$$

where σ_v is the standard deviation of the crosswind velocity component.

To estimate the maximum downwind distance (x), differentiating the above equation with respect to x, setting the result equal to zero, and then solving for x, to get:

$$\frac{\partial C}{\partial x} = \left[\frac{AQ}{\sqrt{2\pi}\,\sigma_v z} \exp\left(-\frac{y^2 \overline{U}^2}{2\sigma_v^2 x^2}\right)\right]\left(\frac{y^2 \overline{U}^2}{\sigma_v^2 x^2} - \frac{1}{x^2}\right) \quad (13)$$

Putting $\dfrac{\partial C}{\partial x} = 0$, then the maximum downwind distance of (x) becomes:

$$\left(\frac{y^2 \overline{U}^2}{\sigma_v^2 x^4} - \frac{1}{x^2} \right) = 0 \Rightarrow \qquad x_{\max} = \frac{y\overline{U}}{\sigma_v} \qquad (14)$$

Substituting from equation (14) in equation (12) to get the maximum ground level concentration "C_{GLCmax}" by putting $y = 0$ as follows:

$$C_{GLC\,max} = \frac{AQ}{\sqrt{2\pi}\,\sigma_v x_{\max} z}\exp\left(-\frac{y^2\overline{U}^2}{2\sigma_v^2 x^2} \right)$$

$$C_{GLC\,max} = \frac{AQ}{\sqrt{2\pi}\,ey\,\overline{U}\,z} \qquad (15)$$

Dividing equation (12) by equation (15) to get the ratio between C_{GLC} and $C_{GLC\,max}$ as follows:

$$\frac{C_{GLC}(z=0)}{C_{GLC\,max}(z=0)} = \left(\frac{\sqrt{e}\,x_{\max}}{x} \right)\left[\exp\left(-\frac{y^2\overline{U}^2}{2\sigma_v^2 x^2} \right) \right] \qquad (16)$$

Fig. 1 Shows the computed centroid, variance, skewness, and kurtosis for surface that are compared with measurements made through wind tunnel in neutral stability (Khurshudyan *et al.* 1981). Recall that skewness equal 0.7 and kurtosis equals 3 for reflected Gaussian distribution. The downwind variations of the measured centroid, variance, skewness and kurtosis agree well with model predictions.

5. Calculating plume advection velocity

The mean plume wind speed $\overline{U}$ is defined as:

$$\overline{U} = \frac{\displaystyle\int_0^\infty U(z)C_y(x,z)dz}{\displaystyle\int_0^\infty C_y(x,z)dz} \qquad (17)$$

Where the wind speed is given as:

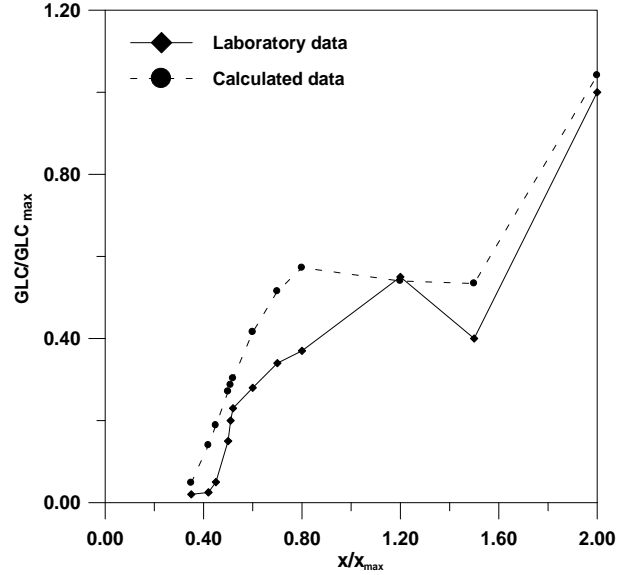$$U(z) = \frac{u_*}{k_v}\left[In\left(\frac{z}{z_0} \right) \right] \qquad (18)$$

Fig. 2. Non–dimensional maximum ground level concentrations as function of non-dimensional downwind distance. Measurements include wind-tunnel experiments

Where u_* is the friction velocity, k_v is von-Kaman constant = 0.4 and z_0 is the roughness height (m). We substitute from equations (5), (18) in equation (17) to get the mean plume wind speed in the boundary layer:

$$\overline{U} = \frac{\Gamma\left(\dfrac{1}{s} \right)In(z) - \Gamma\left(\dfrac{1}{s} \right)In[(z_0)] - \dfrac{1}{s}\Gamma\left(\dfrac{1}{s} - 1 \right)}{\Gamma\left(\dfrac{1}{s} \right)} \qquad (19)$$

Fig. 2 shows a non-dimensional plot of ground level concentration and downwind distance for surface releases from neutral USEPA wind tunnel point source dispersion experiment (Brown *et al.* 1993). The figure illustrates ground level concentration by "GLCmax" [equation (16)] and the downwind distance "$x$max" [equation (14)] have good agreement between the predicted and measured data.

6. Conclusion

Equations of the centroid, variance, skewness, and kurtosis compared well with wind tunnel plume dispersion measurements. The equations derived for the magnitude and location of "GLC_{max}" were found to be agreement well with wind-tunnel measured data. Finally, equation of the plume advection velocity is derived.

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