Mausam, (1985), 36, 2, 185-190

621.746 : 518.61

Optimal control of free-boundary systems applied to continuous casting process*

C. SAGUEZ
INRIA, France

(*Received 2 February 1981*)

सार — एक सतत ढलाई विधि से संबद्ध किसी अबाध सीमा प्रणाली के लिए एक अनुकूलत्तम नियंत्रण निर्मेय के हल की सामान्य विधि प्रस्तुत की गई है। सर्वप्रथम प्रणाली के इन्थाल्पी फलन के उपयोग द्वारा भौतिक निर्मेय सूत्रबद्ध किया जो कि अनुकूलत्तम एकदिष्ट कारक है। योसिदा सन्निकटन के प्रवेश द्वारा हमने आवश्यक अनुकूलत्तम दशांए प्राप्त की। तब हमनें प्रवणता विधि का उपयोग संख्यात्मक रूप से किया। अन्ततः दो निरुद्ध मामलों के लिए कुछ संख्यात्मक परिणाम प्रस्तुत किए गए हैं

ABSTRACT. We present a general method to solve an optimal control problem for a free-boundary system associated with a continuous casting process. Firstly, we formulate the physical problem by using the enthalpy function of the system, which is a maximal monotone operator. By the introduction of the Yosida approximation, we obtain necessary optimality conditions. Then numerically, we use a gradient method. Finally some numerical results are presented for two cases of contraints.

1. Introduction

The expansion of the continuous casting process has been very important since ten years, principally for two reasons : an economic motivation, for this technique the price of the production is less than for the classical method and a metallurgical reason, connected to the quality of the steel. For those facts, it is very interesting to develop methods to optimize the productivity of a such system. Mathematically, we have a free-boundary problem (solidification of the steel) of Stefan type. The numerical methods to solve a such problem have been very extensively studied, in particular by the use of variational inequalities (Duvaut 1976, Lions 1975). In this paper, we formulate the problem have been very extensively studied, in partition, we can study very general problems (non-linear diffusion operator, general boundary conditions, ...). The optimal control problem, we consider, is to determine a control (exchange coefficient between water and steel) to maximize the speed of the cast, with different structural and metallurgical contraints. With simplified problem, we present in this paper a numerical method to compute the optimal control.

In the first part, we explain the physical problem and the mathematical formulation. In a second part, we define, for a semi-discretized problem, necessary optimality conditions and we propose a numerical method based on the gradient algorithm. Finally in a last part, we present some numerical results.

2. The continuous casting process

2.1. *The physical system*

The principle of the continuous casting process is to cast the steel in a mould the bottom of which is constituted by the solidified ingot, which is continuously extracted. The scheme of a such system is given in Fig. 1.

The steel is casted continuously in a copper mould by a nozzle. At the end of the mould, a very small thickness, which is sufficient to avoid break out, is solidified. In a second part, the steel is cooled by a spray system, which is devided in different zones. Each zone is independent. At the end of this part, the steel is cut by a cutting torch.

We distinguish several types of products, following their dimensions : slabs, blooms, billets.

2.2. *The state equation*

To modelize a such system, we consider the following equations. T_1 (resp. T_2) denotes the temperature

*This work has been done, in collaboration with the Research Institute of the French Steel Industry (IRSID) and presented in the symposium Indo-French school on recent advances in Computer Techniques in Meteorology, Biomechanics and applied systems" held at I.I.T., New Delhi, February 1980.

of the steel in the liquid (resp. solid) phase. T_1 verifies the heat equation :

$$\rho V \frac{\partial T_1}{\partial t} - \triangle T_1 = 0 \qquad (2.1)$$

with V the speed of extraction,

c_1 the heat capacity of the liquid,

ρ the density of the steel

In the solid phase, we have similarly the equation :

$$\rho \mathrm{V} \frac{\partial T_2}{\partial t} - \triangle T_2 = 0 \qquad (2.2)$$

with c_2 the heat capacity of the solid.

Along the front of solidification $S(t)$ the free-boundary we have the two conditions :

$$T_1 \mid_{S(t)} = T_2 \mid_{S(t)} = T_S \text{ (T_S temperature of solidification)} \qquad (2.3)$$

$$\mathbf{grad}\ T_1.\mathbf{n} - \mathbf{grad}\ T_2.\mathbf{n} - L\ \mathbf{q}.\mathbf{n} \qquad (2.4)$$

with L the latent heat of solidification

q. n the normal speed of the free-boundary.

Finally, we give the boundary conditions and the initial conditions. To simplify the presentation, we consider the unique boundary condition :

$$\frac{\partial T}{\partial n} + h(T - T_e) \mid_{\Sigma} = 0 \qquad (2.5)$$

with h a exchange coefficient between the steel and the water, T_e the temperature of the water.

In practice in the mould, the flux of heat $\partial T/\partial n$ is given and for the spray-system, we have similar conditions as above. As initial condition, we take :

$$T(x, 0) = T_0(x) \qquad (2.6)$$

3. The optimal control problem

The optimal control problem is to find h (coefficient of exchange) which maximize the speed of extraction such that structural and metallurgical contraints are verified. More precisely, we have the contraints;

(*i*) the quantity of water, we can use, is bounded.

(*ii*) the steel must be completely solidified before the cutting-torch.

(*iii*) at the unbending point, the temperature must verify the condition :

$T \in]T_1, T_2[$

(*iv*) the gradient of the temperature along the boundary is bounded.

In this paper, we don't consider exactly this problem but a sub-problem : to find an admissible control h such that the state of the system verifies constraints, (*i*), (*ii*), (*iii*). So mathematically, we have the following problem. If T denotes the temperature of the solid and

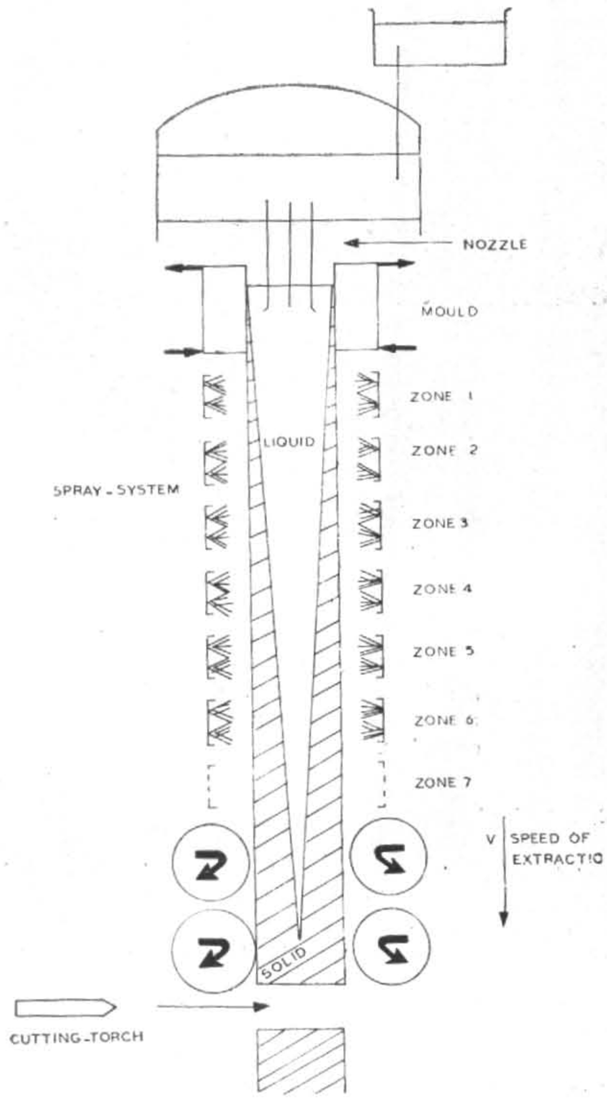

Fig. 1. Continuous casting process

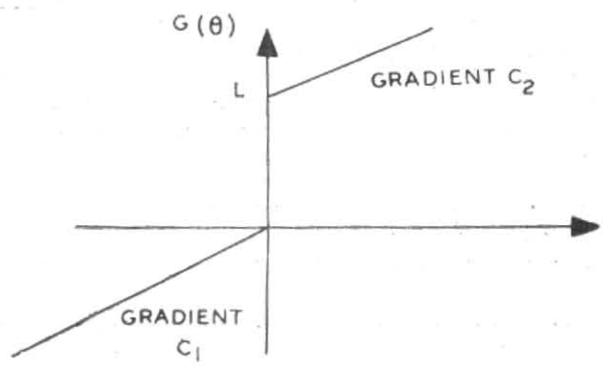

Fig. 2

the liquid and if $\theta = T_S - T$, it is easy to prove that θ is solution of the system :

$$\begin{cases} \rho V \dfrac{\partial u}{\partial t} - \triangle\theta = 0 \ ; u \in G(\theta) & (3.1) \\ \dfrac{\partial\theta}{\partial n} + h(\theta - \theta_e) \mid_{\Sigma} = 0 & (3.2) \\ \theta(x,0) = \theta_0(x);\ u(x,0) = u_0(x) \in G(\theta_0) & (3.3) \end{cases}$$

with $\theta_e = T_S - T_e$ and $G(\theta) = H(T_S - \theta)$ (H enthalpy for the system). G is a maximal monotone operator, the graph of which is given in Fig. 2.

We assume that :

$$h = \sum_{i=1}^{N_Z} h_i X_{E_i},\ h_i \in \mathbf{R}^+ \quad (3.4)$$

with N_Z number of zones of the spray-system. X_{E_i} characteristic function of the zone E_i.

Then we introduce the functional :

$$J(h) = \mid\mid (u - \mathrm{L})^- \mid\mid^2_{L^2(Q)} \quad (3.5)$$

We consider two types of contraints :

Case 1

$$\begin{cases} 0 < h_{im} \leqslant h_i \leqslant h_{iM} \\ \displaystyle\sum_{i=1}^{N_Z} \alpha_i h_i \leqslant D_M \end{cases} \quad (3.6)$$

Case 2

$$\begin{cases} 0 < h_{im} \leqslant h_i \leqslant h_{iM} \\ \theta(x, t_1) \leqslant \theta_1 \end{cases} \quad (3.7)$$

which correspond respectively to the constraints (*ii*) and (*iii*). So the optimal control problem is to find $\bar{h}$ in the admissible set of controls U_{ad} such that :

$$J(\bar{h}) \leqslant J(h) \qquad \forall\ h \in U_{ad} \quad (3.8)$$

Remark 2.1

The problem is, in fact, a particular case of a general optimal control problem of a Stefan system. So the method, we present in the following part, is general for all problems of this type.

4. Numerical method

4.1. *Transformation of the state equation*

Following a result of a Bermudez. & Moreno (1978), we have the equivalence :

$$u \in G(\theta) \Longleftrightarrow u = G_\mu(\theta + \mu u) \quad (4.1)$$

where G_μ is the Yosida approximation of G.

So, if we introduce $G_\omega(\theta) = G(\theta) - \dfrac{\omega}{\rho V}\theta$, with $0 < \omega < \rho V$ Min (c_1, c_2) and $\overset{\mu}{\underset{\omega}{G}}$ the Yosida approximation of G_ω ($\overset{\mu}{\underset{\omega}{G}}$ is a strictly monotone operator), can write the state equation (3.2)-(3.3) as :

$$\begin{cases} \rho V \dfrac{\partial w}{\partial t} - \triangle\theta + \omega\theta = 0 \ ; w \in G^{\mu}_{\theta}(\theta + \mu w) & (4.2) \\ \dfrac{\partial\theta}{\partial n} + h(\theta - \theta_e) \Big|_{\Sigma} = 0 & (4.3) \\ \theta(x,0) = \theta_0(x), w(x,0) = w_0(x) = \\ \qquad u_0 - \dfrac{\omega}{\rho V}\theta_0 \in G_\omega(\theta_0) & (4.4) \end{cases}$$

Then we introduce the system, semi-discretized by an implicit scheme :

$$\begin{cases} \rho V \left(\dfrac{W^{n+1} - W^n}{k}, \psi\right) + \dfrac{\omega}{k}(\theta^{n+1} - \theta^n, \psi), + \\ (\operatorname{grad}\theta^{n+1}, \operatorname{grad}\psi) + \displaystyle\int_\Gamma h^{n+1}(\theta^{n+1} - \theta_e^{n+1})\psi d\Gamma = 0 \\ \qquad \forall \psi \in H^1(\Omega) & (4.5) \\ W^{n+1} \in G_\omega(\theta^{n+1}) = G(\theta^{n+1}) - \dfrac{\omega}{\rho V}\theta^{n+1} \\ \qquad n = 0, \ldots, NT-1 & (4.6) \\ \theta^0 = \theta_0(x), W^0 = u_0 - \dfrac{\omega}{\rho V}\theta_0 & (4.7) \end{cases}$$

For the functional J, we take :

$$J(h) = \sum_{n=1}^{NT} \Big|\Big| (u^n - \mathrm{L})^- \Big|\Big|^2_{L^2(\Omega)} \quad (4.8)$$

4.2. *Resolution of the state equation*

To solve in (W^{n+1}, θ^{n+1}) the system (4.5)-(4.6), we propose the following algorithm. At each step $(n+1)$, we do the iterations [(W^n, θ^n) is given)] :

$$\frac{\omega}{k}(\theta_j^{n+1}, \psi) + (\operatorname{grad}\theta_j^{n+1}, \operatorname{grad}\psi) + \\ + \int_\Gamma h^{n+1}(\theta_j^{n+1} - \theta_e^{n+1})\ \psi\, d\Gamma = -\frac{\rho V}{k}(W^{n+1} \\ - W^n, \psi) + \frac{\omega}{k}(\theta^n, \psi) \quad \forall\ \psi (H^1(\Omega)) \quad (4.9)$$

$$W^{n+1}_{j+1} = \overset{\mu}{\underset{\omega}{G}}(\theta_j^{n+1} + \mu W_j^{n+1}) \quad (4.10)$$

Eqn. (4.10) is well-defined with the properties of $\overset{\mu}{\underset{\omega}{G}}$. We have the result of convergence

Proposition 1

for $\mu > \mu_0 = \dfrac{\rho V}{k} \times \dfrac{1}{2 \text{ Min } 1, \left(\dfrac{\omega}{k}\right)}$, we have,

$W_j^{n+1} \longrightarrow W^{n+1}$ in $L^2(\Omega)$ weakly

$\theta_j^{n+1} \longrightarrow \theta^{n+1}$ in $H^1(\Omega)$ strongly

Demonstration

We know that $\overset{\mu}{\underset{\omega}{G}}$ is a Lipschitzian operator; so we obtain :

$$\Big|\Big| W^{n+1} - W^{n+1}_{j+1} \Big|\Big|^2_{L^2} \leqslant \frac{1}{\mu^2} \Big|\Big| \theta^{n+1} + \\ + \mu W^{n+1} - \theta_j^{n+1} - \mu W_j^{n+1} \Big|\Big|^2_{L^2} \quad (4.11)$$

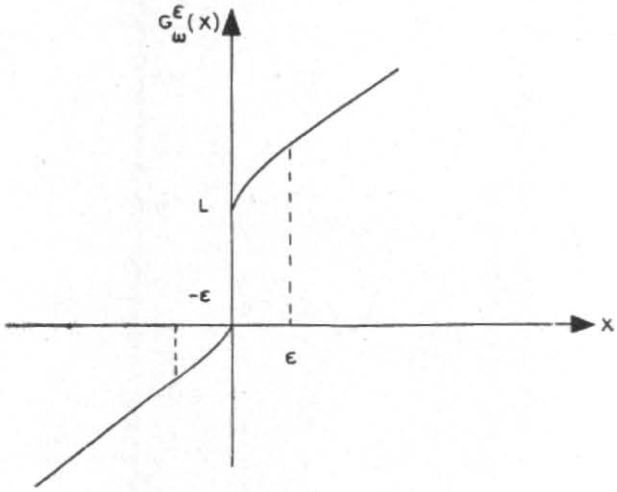

Fig. 3

$$\leqslant \left|\left| W^{n+1} - W_j^{n+1} \right|\right|^2_{L^2} + \frac{2}{\mu}(\theta^{n+1} - \theta_j^{n+1}, W^{n+1} - W_j^{n+1}) + \frac{1}{\mu^2}\left|\left|\theta^{n+1} - \theta_j^{n+1}\right|\right|^2_{L^2} \qquad (4.11)$$

From (4.9), we obtain :

$$\left|\left|\text{grad}\,(\theta^{n+1}-\theta_j^{n+1})\right|\right|^2_{L^2} + \frac{\omega}{k}\left|\left|\theta^{n+1}-\theta_j^{n+1}\right|\right|^2_{L^2} + \frac{\rho V}{k}(W^{n+1}-W_j^{n+1}, \theta^{n+1}-\theta_j^{n+1}) + \int_\Gamma h\,(\theta^{n+1}-\theta_j^{n+1})^2\, d\Gamma = 0 \qquad (4.12)$$

Then $\frac{\rho V}{k}(W^{n+1}, W_j^{n+1}, \theta^{n+1}-\theta_j^{n+1})$

$$\leqslant -\text{Min}\left(1, \frac{\omega}{k}\right)\left|\left|\theta^{n+1}-\theta_j^{n+1}\right|\right|^2_{H^1(\Omega)}$$

and from (4.11)-(4.12) :

$$\left|\left|W^{n+1}-W_j^{n+1}\right|\right|^2_{L^2(\Omega)} \leqslant \left|\left|W^{n+1}-W_j^{n+1}\right|\right|^2_{L^2(\Omega)} + K\left|\left|\theta^{n+1}-\theta_j^{n+1}\right|\right|^2_{H^1(\Omega)} \qquad (4.13)$$

with $K = \frac{1}{\mu^2} - \frac{2}{\mu}\, k\, \frac{\text{Min}\,(1, \frac{\omega}{k})}{\rho V} < 0$ for $\mu > \mu_0$

So with $\mu > \mu_0$, $\left|\left|W^{n+1} - W^{n+1}_{j+1}\right|\right|^2_{L^2(\Omega)}$ is a bounded decreasing sequence in **R**.

Then from (4.13) we deduce that :

$$\left|\left|\theta^{n+1}-\theta_j^{n+1}\right|\right|_{H^1(\Omega)} \longrightarrow 0$$

TABLE 1

Conductibility $\lambda\,(T)$

T(°C)	λ(cal/cm/s/°C)	T (°C)	λ(cal/cm/s/°C)
550	0,091	1100	0,068
600	0,086	1150	0,070
650	0,081	1200	0,071
700	0,076	1250	0,072
750	0,071	1300	0,073
800	0,068	1350	0,075
850	0,065	1400	0,076
900	0,064	1450	0,077
950	0,065	1500	0,078
1000	0,066	1550	0,079
1050	0,067	1600	0,080

and from (4.9), we conclude that

$W_j^{n+1} \longrightarrow W^{n+1}$ in $L^2(\Omega)$ weekly.

4.3. *Determination of the optimal control*

In fact we compute the optimal control of a regularized problem, for which we can prove the existence of optimality conditions. So a gradient method can be used.

For this, we regularize G_ω by $\overset{\epsilon}{G}_\omega$, the graph of which is given in Fig. 3, such that $\overset{\epsilon}{G}_\omega$ is the sub-differential of a strictly convex, l.s.c., proper function $\overset{\epsilon}{\psi}_\omega$. We suppose that $G^{\epsilon,\mu}_\omega \in C^2(\mathbf{R})$ and that

$$|\psi^\epsilon_\omega(x) - \psi_\omega(x)| \leqslant k(\epsilon)\,|x| \text{ with } \begin{cases} k(\epsilon) \to 0 \\ \quad \epsilon \to 0 \end{cases}$$

Then, the regularized problem is the initial system in which we have replaced $W \in G_\omega(\theta)$ by $W \in G_\omega^\epsilon(\theta)$. So if we consider the optimal control problem associated with this regularized state equation, we prove the existence of an optimal control h_ϵ and that the sequence $\{h_\epsilon\}$ converges to be solution of the initial problem. But the main result is that we have the following optimality conditions :

State equation

$$\begin{cases} \frac{\rho V}{k}(W_\epsilon^{n+1} - W_\epsilon^{n}, \psi) + (\text{grad}\,\theta_\epsilon^{n+1}, \text{grad}\,\psi) + \frac{\omega}{k}(\theta_\epsilon^{n+1} - \theta_\epsilon^{n}, \psi) + \int_\Gamma h_\epsilon^{n+1}(\theta_\epsilon^{n+1} - \theta_\epsilon^{n+1})\,\psi\, d\Gamma = 0 \quad \forall \psi \in H^1(\Omega) & (4.14) \\ W_\epsilon^{n+1} = G^{\epsilon,\mu}_\omega(\theta_\epsilon^{n+1} + \mu W_\epsilon^{n+1}) \quad n = 0, \ldots NT-1 & (4.15) \\ \theta_\epsilon^0 = \theta_0\,;\; W_\epsilon^0 = W_0^\epsilon \in G^\epsilon_\omega(\theta_0) & (4.16) \end{cases}$$

Adjoint state equation

$$\begin{cases} \frac{\rho V}{k}\left\{ q_\epsilon^n \, G_\omega^{\epsilon,\mu} (\theta_\epsilon^{n+1} + \mu W_\epsilon^{n+1}), \psi \right\} + (\text{grad } p_\epsilon^n, \text{grad } \psi) + \\ + \frac{\omega}{k} (p_\epsilon^n - p_\epsilon^{n+1}, \psi) + \\ + \int_\Gamma h_\epsilon^{n+1} p_\epsilon^n \, \psi d\,\Gamma = -2 \left\{ \left(u_\epsilon^{n+1} - L\right)^-, \psi \right\} \\ \forall\, \psi \in H^1(\Omega) \qquad (4.17) \\ p_\epsilon^n - p_\epsilon^{n+1} = q_\epsilon^n \left\{ 1 - \mu\, G_w^{\prime \epsilon,\mu} (\theta_\epsilon^{n+1} + \mu\, W_\epsilon^{n+1}) \right\} \\ n=0,\ldots,NT-1 \qquad (4.18) \\ p_\epsilon^{NT} = 0 \qquad (4.19) \end{cases}$$

Optimality conditions

$$\sum_{n=1}^{NT} \int_\Gamma (\theta_\epsilon^n - \theta_e^n)\, p_\epsilon^{n-1}\, (s^n - h_\epsilon^n) d\,\Gamma \leqslant 0 \quad \forall \{s^n\}_\epsilon \; U_{ad} \Box \tag{4.20}$$

The demonstration of this result is given in Saguez (*loc. cit.*). We prove the G-differentiability of θ_ϵ and W_ϵ with respect to h and so, by the introduction of a well chosen adjoint state, we obtain the condition (4.20).

Remark 4.1

It is important to see that G_w^ϵ is still a multivoque operator. Then by this method we obtain optimality conditions when $m\ \{(x,t)\ |\ \theta(x,t)=0\}>0$. So we remark that this method is very general. □

With the above results, we can compute the optimal control by a gradient method. For the case 1 (constraints on the control), we use the following algorithm introduced by Henry (1978). At each iteration, if we denote by $\{h_i^l\}$ the last control and by $\{G_i^l\}$ the associated gradient, we obtain $\{h_i^{l+1}\}$ by the iterations :

$$\begin{aligned} h_i^{l+1,j} &= \text{Max}\left[h_{im}, \text{Min}\left\{ h_{iM}, h_i^l - \rho_l^j (G_i^l - \lambda^j \alpha_i) \right\} \right] \\ \lambda^{j+1} &= \text{Max}(0) \left[\lambda^j - \rho_2^j \left(\sum_i \alpha_i h_i^{l+1,j} - D \right) \right] \end{aligned} \tag{4.21}$$

In fact, this algorithm corresponds to a gradient method with projection. Its implementation is very easy.

For the case 2 (constraint on the sate); we use a penatly method.

5. Numerical results

5.1. *The data*

We have solved the problem for a real continuous casting. So the state equation is not exactly the same than in the Chapter 3. In particular, the conductibility

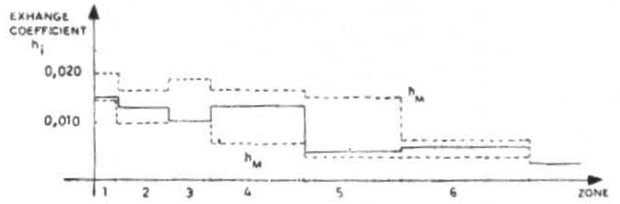

Fig. 4. Optimal control

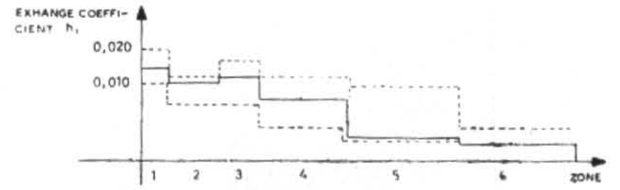

Fig. 5. Optimal control

λ is a function of the temperature (*see* Table 1). We have considered the one-dimensional case. The state T is solution of the system :

$$\rho V \frac{\partial v}{\partial t} - d\omega\,[\lambda(T) \text{ grad } T] = 0\ ;\ v \in H(T) \text{ for } x \in]0, R[\ ;\ t \in]0, T[\tag{5.1}$$

In the mould, we have the boundary condition :

$$\lambda(T).\ \textbf{grad}\ T.\textbf{n} = g \quad \text{for } x=0 \tag{5.2}$$

For the spray system, we have :

$$\lambda(T)\ \textbf{grad}\ T\ .\ \textbf{n} = -h(T - T_e) \text{ for } x=0 \tag{5.3}$$

And for $x=R$, the condition is :

$$\lambda(T)\ \textbf{grad}\ T.\textbf{n} = 0 \tag{5.4}$$

The initial condition is always :

$$T(x,0) = T_0(x)\ ;\ v(x,0) = v_0(x) \in H(T_0). \tag{5.5}$$

With these equations we have the following data :

Structural data :

—Length of the mould 70 cm

—Spray-system :

zone 1	60 cm
zone 2	200 cm
zone 3	150 cm
zone 4	350 cm
zone 5	390 cm
zone 6	430 cm
zone 7	1250 cm

Total length of the installation 29 m.

$R=12,5$ cm (which corresponds to a slab)

Physical Data

$c_1=0{,}2$ cal/g/°C; $c_2=0{,}16$ cal/g/°C

$L=59{,}09$ cal/g; $Ts=1502$ °C

Density of the steel $\rho=7$g/cm³

Initial temperature $T_0=1537$ °C

The step of discretization is $k=10$ cm, $\triangle x=12{,}5/40$ cm.

TABLE 2
Flux in the mould

mould (cm)	flux (cal/cm²/s)
10	20,70
20	48,98
30	45,46
40	37,59
50	32,36
60	26,28
70	26,89

5.2. *Numerical results*

Case 1

$$\begin{cases} h_{im} \leqslant h_i \leqslant h_{iM} \\ \sum_{i=1}^{7} \alpha_i h_i = D \end{cases} \quad ; V=1{,}50 \text{ m/mn.}$$

We take the data (Table 3)

TABLE 3

zone	h_{im}	h_{iM}	α
1	0,015	0,020	0,40
2	0,010	0,016	0,5790
3	0,010	0,018	0,4902
4	0,0060	0,0157	0,8285
5	0,0030	0,0144	1,0492
6	0,0030	0,0066	0,9125
7	0,0025	0,025	0

We obtain the results :

Zone	1	2	3	4	5	6
h_i (cal/cm² s° C)	0,015	0,0128	0,010	0,0132	0,00439	0,00483

Case 2

$$\begin{cases} h_{im} \leqslant h_i \leqslant h_{iM} \\ \theta(x,t_1) \leqslant \theta_1 \end{cases} \qquad V=0{,}9\text{m/mm}$$

with $\begin{cases} \theta_1 = 502\ °C \\ t_1 = 16\ m \end{cases}$

and we consider only the first six zones of the spray-system. For a penalty coefficient $\epsilon = 2 \times 10^{-1}$, we obtain the optimal control :

Zone	1	2	3	4	5	6
h_i(cal/cm² s°C)	0,01858	0,01544	0,01576	0,01146	0,00313	0,003

In each case, the computer time is approximately equal to 2 mn on IBM 370-168.

The above method has been very efficient to solve the studied problem. The next step of this work, is to use this method to develop a system of regulation for a continuous casting process. Also this method is general, and can be used for many other multiphase systems.

References

Alberny, R., 1978, "Presentation de ta coulée continue", (Rapport Interne IRSID, Nov 1978).

Alberny, R., Perroy A., Amory, D. and Lahosse, M., 1978, "Optimisation du refroidisement second air en coulée continue de brames d' acier extra-doux (Revue de metallurgie, Juin 1978.

Bermudez, A. and Moreno, C., 1978, "Duality methods for solving variational inequalities".

Bonnans, F., 1979, "Contrôle optimal d'un probleme a frontiere libre", (Stage IRIA 1979).

Brezis, H., 1973, "Opérateurs maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert", (North Holland Publishing Company).

Degueil, A., 1977, "Resolution par une méthode d'éléments finis d'un problème de Stefan en termes de température et teneur en matériau non gelé"(Thèse 1977)

Duvaut, G., 1976, "Problèmes á frontière libre en théorie des milieux continus" (Rapport LABORIA N° 185).

Grange, O. and Mignot, F., 1972, "Sur la résolution d'une équation et d'une inéquation parabolique d'évolution", *J. of Functional Analysis.*

Henry, J., 1978, "Contrôle d'un réacteur enzymatique á l'aide de modéles ápáramètres distribués. Quelques problemes de Controlabilité de systémes paraboliques" (Thèse 1978).

Larrecq, M., Saguez, C. and Wanin, M., 1978 "Modèle mathématique de la solidification en coulée continue en tenant compte de la convection á l'interface solide-liquide", Revue de métallurgie Juin 1978.

Lions, J.L., 1975, "Introduction to some aspects of free boundary problems", Colloque Synspade Maryland.

Saguez, C., "Controle optimal dum systeme a frontiere libre, application a l'optimisation du refroidissement secondaire en coulee continued' acier (Rapport Laboria a paraitre).

Sauez, C. and Larreq, M., 1979, "Controle de systemes a frontiere libre, application a la coulee continiue d'acier."