Mausam, (1985), 36, 2, 191-196

517.93

Catastrophe theory*

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(Received 2 February 1981)

सार — वर्तमान शोध पत्र में अरेखीय अवकलीय प्रणाली के अध्ययन के लिये थोम द्वारा प्रतिस्थापित विध्वंस सिद्धांत की चर्चा की गई है। इसमें पूर्ववर्ती लेखकों द्वारा किए गए पहले वर्गीकरण के अनुसार सात प्राथमिक विध्वंस घटनाएं प्रस्तुत की गई हैं। डफिंग के समीकरण के अरेखीय क्रम में वृद्धि करते हुए हमने समीकरण $\ddot{x} + 2\zeta \dot{x} + x + \alpha x^3 + \beta x^5 = \phi \cos \Omega t$ का अध्ययन किया है। उक्त मामले में विध्वंस बहुमुखी और तितली के आकार का होता है। जीमन द्वारा प्रयुक्त मस्तिष्क की गति के अध्ययन के लिये डफिंग समीकरण की एक मॉडल के रूप में भी चर्चा की गई है। इस शोध पत्र मे जीमन द्वारा हृदय स्पन्दनों तथा नाड़ी की गति के विषय में दिए गए अवकलीय समीकरण पर भी चर्चा की गई है। इस शोध पत्र म विध्वंस सिद्धांत तथा इसके अनुप्रयोगों की भिन्नताओं पर भी संक्षेप में चर्चा की गई है। ऐसा अनुभव किया जा रहा है कि विध्वंस सिद्धान्त का गहन अध्ययन किया जाना चाहिए क्योंकि इससे हमें अपनी वैज्ञानिक तथा तकनीकी गतिविधियों को अधिक निकट से देखने का अवसर मिलेगा।

ABSTRACT. The catastrophe theory, to study the non-lnear differential system, as enunciated by Thom is discussed. Seven elementary catastrophes according to the classifications made by earlier authors are presented here. Increasing the order of non-linearity of the Duffing's equations we have studied the equation $\ddot{x} + 2\zeta\dot{x} + x + \alpha x^3 + \beta x^5 = \phi \cos \Omega t$.
The catastrophe manifold in this case is butterfly catastrophe. The Duffing's equation is also discussed as a model to study the dynamics of brain used earlier by Zeeman. The differential equation for heart beat and nerve impulse given by Zeeman are also discussed in this paper. The controversies regarding catastrophe theory and its variety of applications are summarized.

1. Introduction

Mostly a mathematical model of physical or dynamical system is represented by a set of state and control variables satisfying a set of differential equations (Ames 1968). In the formulation of such a mathematical model there is the underlying assumption that the state of a given system changes smoothly if the cause of its variation is small. It means a small effect should result into a small variation in the variables determining the state of a structurally stable system. But many systems particularly those represented by nonlinear differential equations may suddenly change due to smooth alteration in the situation. Many such problems have been analysed with variety of mathematical methods (Hale 1969). To analyse a broad range of such phenomena in a coherent manner, Thom (1975) has developed a mathematical technique and called it catastrophe. This is now known as catastrophe theory (Poston and Stewart 1978) and should be viewed as a new development within calculus.

Stability lies at the root of the modern mathematical theories of dynamical systems and singularities. It is important to note that small forces or alterations in the situation, can change the state of a dynamical system suddenly only when it is at a certain critical state in the process of its evolution. So it is not only important to determine the state of a dynamic system but also to find the critical points in its history at which a catastrophic change may occur due to small changes in the controlled parameters.

Structural stability in certain sense can be treated as catastrophe theory and so can be studied by minimization of certain function usually called potential.

The theory of structural stability becomes involved when several variables (state and control) are required to determine the dynamics of the system. In such a situation catastrophe theory can be effectively used to determine the critical points and also behaviour of the system at those points. Let us start with a family of functions :

$$V : S \times C \rightarrow R$$

S being state variable manifold R^n and C control variable manifold R^r, say. The catastrophe manifold M is the subset of $R^n \times R^r$ defined by

$$\frac{dV(x,c)}{dx} = 0$$

This is the set of all critical point of $V(x, c)$.

The catastrophe map χ is the restriction to M on the natural projection :

$$F : R^n \times R^r \rightarrow R^r$$

*The paper was presented in the symposium "Indo-French School on recent advances in Computer Techniques in Meteorology, Bio-mechanics and applied systems "held at I.I.T., New Delhi, February 1980.

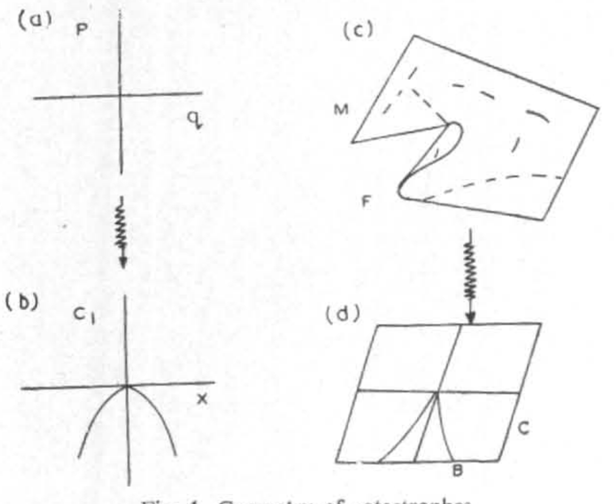

Fig. 1. Geometry of catastrophes

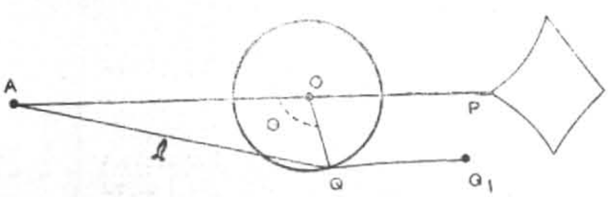

Fig. 2. Mechanical model of Zeeman catastrophe machine

for which

$$F(x, c) = C_0$$

The singularity set S is the set of singular points in M of χ. The image $\chi(S)$ in C is called the bifurcation set B, on which the number and nature of the critical points change.

The catastrophe theory classifies the singularities of V, that are structurally stable. It means that when a system is subjected to arbitrary perturbation, unfolding is topologically unchanged (Berry 1977).

Here the first seven catastrophe geometries are given. These ideas are then used to analyse Duffing's equation after making certain modification in the nonlinear term. Zeeman (1977) in one of his papers used Duffing's equation in brain modelling. Catastrophe theory is also applied by Zeeman to heart beat and nerve impulses (Zeeman 1973). These two papers are also discussed in this note.

2. Seven elementary catastrophes

Let us consider a system whose behaviour can be described by a finite number of state variables x_1, x_2........and finite number of control variables C_1, C_2......., say. Thom had shown that any such system, by continuously changing its control variables in the neighbourhood of a given state, can exhibit catastrophic jump behaviour. For a system of codimension $\leqslant 4$, there are seven possible catastrophes (Table I). Only these catastrophes are considered here. For higher catastrophes one may refer to the work of Poston *et al.* 1978).

TABLE 1

Catastrophe	Energy function, V
Fold	$(1/3)x_1^3 + C_1 x_1$
Cusp	$(1/4) x_1^4 + (1/2) C_1 x_1^2 + C_2 x_1$
Swallowtail	$(1/5) x^5_1 + (1/3) C_1 x_1^3 + +(1/2) C_2 x_1^2 + C_3 x_1$
Butterfly	$(1/6) x_1^6 + (1/4) C_1 x_1^4 + (1/3) C_2 x_1^3 + + (1/2) C_3 x_1^2 + C_4 x_1$
Elliptic Umbilic	$x^3_1 - 3x_1 x_2^2 + C_1 x_1 + +C_2 x_2 + C_3 (x_1^2 + x_2^2)$
Hyperbolic Umbilic	$x_1^3 + x_2^3 + C_1 x_1 + C_2 x_2 + + C_3 x_1 x_2$
Parabolic Umbilic	$x_2 x_1^2 + x_2^4 + C_1 x_1 + C_2 x_2 + + C_3 x_1^2 + C_4 x_2^2$

The general procedure to determine the geometry of of these catastrophes is already given earlier. Here the details of this procedure are given for cusp catastrophe.

Energy function V for cusp catastrophe is

$$V(x_1, C_1, C_2) = (1/4)x_1^4 + (1/2) C_1 x_1^2 + C_2 x_1 \quad (2.1)$$

The catastrophe manifold M is given by

$$\frac{dV}{dx_1} = x_1^3 + C_1 x_1 + C_2 = 0 \quad (2.2)$$

Using (2.2) we can use the two variables (x_1, C_1) as chart on M having general point

$$(x_1, C_1, C_2) = (x_1, C_1, -C_1 x_1 - x_1^3) \quad (2.3)$$

Now the Taylor expansion in the neighbourhood of the state satisfying (2.2), is given as

$$V(x_1 + x) = -(3/4)x_1^4 - (1/2)C_1 x_1^2 + \frac{0}{1} x + + \frac{(3x_1^2 + C_1)}{2} x^2 + x_1 x^3 + (1/4)x^4 \quad (2.4)$$

Putting $p = 3x_1^2/2 + (1/2)C_1$, $q = x_1$, $r = 1/4$, we can now use the plane $r = 1/4$ as chart on M, with

$$(C_1, x_1) = (2p - 3q^2, q)$$

expressing the old chart in terms of new one. The line $p=0$ representing q axis is transformed into fold curve $C_1 = -3x_1^2$, in M. For $p<0$, V has local maximium and for $p>0$. it has local minimum. using this value of C_1 in (2.3) we get the fold line parameterized by x_1 as

$$(x_1, -3x_1^2, 2x_1^3) = (x_1, C_1, C_2) \quad (2.5)$$

Bifurcation set is the image of this in C. This gives

$$4C_1^3 + 27 C_2^2 = 0 \quad (2.6)$$

This geometry is given in Fig. 1.

The cusp catastrophe can be easily demonstrated by Zeeman catastrophe machine. This mechanical model is presented here (Fig. 2).

In this model θ is the single state variable x_1 and coordinates of the point Q_1 referred to O, the centre of the disc as origin, are the control variables (C_1, C_2). By moving the point Q_1 slowly on the plane containing the disc, we observe that at certain state of motion the disc suddenly jumps from one equilibrium state to another. Also by reversing the path of controls, the path in the state space is not necessarily reversed (hysteresis). The slight differences in the path may produce large differences in state (divergence). In a similar manner it is possible to investigate the geometry of the other catastrophes mentioned here (Poston and Stewart 1978).

3. Duffing's equation

With damping and harmonic forcing Duffing equation is

$$\ddot{x}+2\zeta\dot{x}+x+\alpha x^3 = \phi\cos\Omega t \tag{3.1}$$

This equation is studied by Holmes and Rand (1976) for small values of ζ and ϕ. Following their analysis we can write :

$$\dot{x}=y$$

$$\dot{y}=-x-2\zeta y-\alpha x^3+\phi\cos\Omega t \tag{3.2}$$

Using the transformation :

$$u=x\cos\Omega t-\left(\frac{y}{\Omega}\right)\sin\Omega t;\ v=-x\sin\Omega t-\left(\frac{y}{\Omega}\right)\cos\Omega t \tag{3.3}$$

we get

$$\dot{u}=(\rho x+2\zeta y+\alpha x^3-\phi\cos\Omega t)\frac{\sin\Omega t}{\Omega}$$

$$\dot{v}=(\rho x+2\zeta y+\alpha x^3-\phi\cos\Omega t)\frac{\cos\Omega t}{\Omega} \tag{3.4}$$

where $\rho=(1-\Omega^2)$. A solution of the form $A\cos(\Omega t+\psi)$ of (3.1) gives

$$u=A\cos\psi \text{ and } v=A\sin\psi \tag{3.5}$$

For small values of $(\zeta, \alpha, \phi, \rho)$ nearly harmonic solutions of (3.1) can be obtained and then corresponding periodic solutions of (3.4) are nearly constant. Under these conditions the averaged equations are :

$$\dot{u}=-\frac{1}{2\Omega}\left[\rho v+\frac{3\alpha}{4}(u^2+v^2)v+2\zeta\Omega u\right]$$

$$\dot{v}=-\frac{1}{2\Omega}\left[-\rho u-\frac{3\alpha}{4}(u^2+v^2)u+2\zeta\Omega v+\phi\right]$$

Since (3.4) has constant solution we have

$$\phi^2=4\zeta^2\Omega^2A^2+A^2\left[\rho+\left(\frac{3\alpha}{4}\right)A^2\right]^2 \tag{3.6}$$

and $\sin\psi=-(2\zeta\Omega A/\phi)$

This can be compared with

$$\frac{dV}{dx_1}=x_1^3+C_1x_1+C_2=0 \tag{3.7}$$

where, $x_1=A^2+(8\rho/9\alpha)$, $C_1=(16/27\alpha^2)(12\Omega^2\zeta^2-5\rho^2)$

and $C_2=-(16/729\alpha^3)[8\rho(\rho^2+36\Omega^2\zeta^2)+81\alpha\phi^2]$

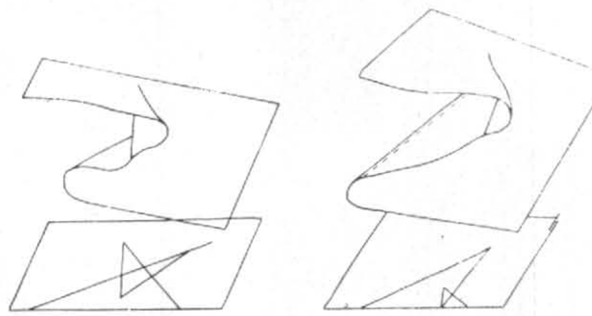

Fig. 3. Some typical cases of catastrophe manifold alongwith the bifurcation set

The solution of Duffing's with small damping and small forcing function exhibits the characteristics of cusp catastrophe.

Now by increasing the order of non-linearity to x^5 in Duffings' equation we have the equation:

$$\ddot{x}+2\zeta\dot{x}+x+\alpha x^3+\beta x^5=\phi\cos\Omega t \tag{3.8}$$

Again using the transformation (3.3) and adopting the earlier procedure, we get

$$\dot{u}=-\frac{1}{2\Omega}\left[\left\{\rho+\frac{3\alpha}{4}(u^2+v^2)+\frac{5\beta}{8}(u^2+v^2)^2\right\}v+2\zeta\Omega u\right]$$

$$\dot{v}=-\frac{1}{2\Omega}\left[-\left\{\rho+\frac{3\alpha}{4}(u^2+v^2)+\frac{5\beta}{8}(u^2+v^2)^2\right\}u+2\zeta\Omega v+\phi\right] \tag{3.9}$$

For constant solution this gives

$$\phi^2=4\zeta^2\Omega^2A^2+A^2\left[\rho+\frac{3\alpha}{4}A^2+\frac{5\beta}{8}A^4\right]^2 \tag{3.10}$$

and $\sin\psi=-(2\zeta\Omega A/\phi)$

The Eqn. (3.10) in amplitude A^2 can be written as :

$$f(x)=x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0 \tag{3.11}$$

where, $x=A^2$, $a_1=(12\alpha/5\beta)$, $a_2=(36\alpha^2+80\rho\beta)/25\beta^2$, $a_3=(96\rho\alpha/25\beta^2)$, $a_4=(64/25\beta^2)(\rho^2+4\zeta^2\Omega^2)$, $a_5=-(64\phi^2/25\beta^2)$

Putting $x=(x_1+h)$ in (3.11) we can write it as

$$f(h)+\frac{f'(h)}{1!}x_1+\frac{f''(h)}{2!}x_1^2+\frac{f'''(h)}{3!}x_1^3+\frac{f^{iv}(h)}{4!}x_1^4+\frac{f^{v}(h)}{5!}x_1^5=0$$

Now choosing $\frac{f^{iv}(h)}{4!}=5h+a_1=0$ and noting that

$$\frac{f^{v}(h)}{5!}=1,$$

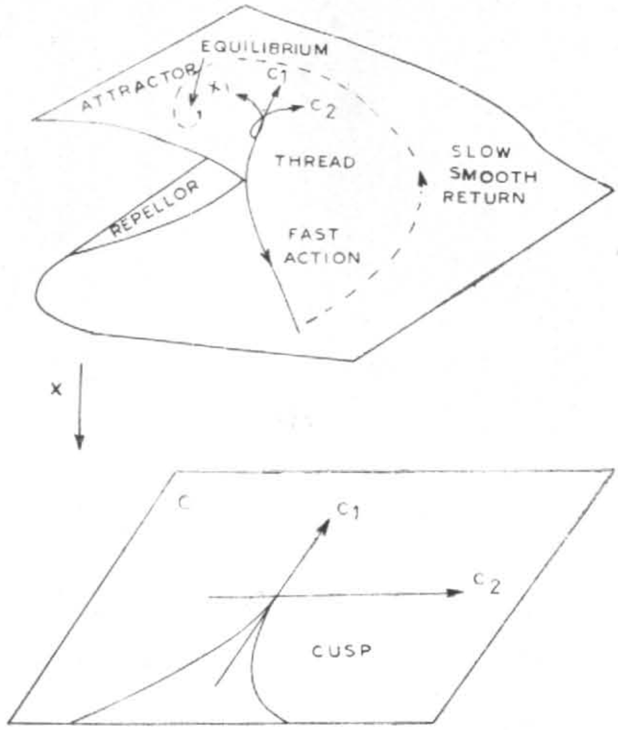

Fig. 4. Projection of fold curves

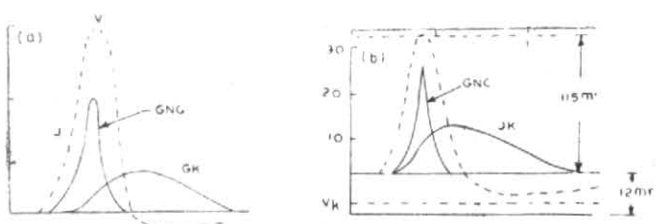

Fig. 5. (a) Theoretical solution for propagated action potential and conductances and (b) Analogues results for the Hodgkin-Huxlay equations

we get

$$\frac{dV}{dx_1} = x_1^5 + C_2x_1^3 + C_1x_1^2 + C_3x_1 + C_4 = 0 \quad (3.12)$$

where, $C_1 = -(1/5)(2a_1^2 - 5a_2)$,

$C_2 = (1/25)(4a_1^3 - 15a_1a_2 + 25a_3)$,

$C_3 = -(1/125)(3a_1^4 - 15a_1^2a_2 + 75a_1a_3 - 125a_4)$,

$C_4 = (1/3125)(4a_1^5 - 25a_1^3a_2 + 125a_1^2a_3 - 625a_1a_4 + 3125a_5)$.

Eqn. (3.12) is the catastrophe manifold M of butterfly catastrophe having energy function

$$V = (1/6)x_1^6 + (1/4)C_1x_1^4 + (1/3)C_2x_1^3 + (1/2)C_3x_1^2 + C_4x_1 \quad (3.13)$$

Again we can obtain Taylor series and quadratic cubic, quartic and quintic terms in it can be taken as coordinates in space R^4. Here the catastrophe manifold maps into C in a fairly complicated manner. The catastrophe manifold alongwith the bifurcation set is shown here for some typical cases (Fig. 3).

4. Duffing's equation and brain modelling

To link the gap between neurology and psychology we need a mathematical approach to describe the medium scale dynamics of brain. Most obvious tool for this purpose according to Zeeman (1977) is to use differential dynamical systems. He has used Duffing's equation in an attempt to explain most obvious feature, the oscillatory nature of brain (Morrison 1979).

(*i*) Most sense organs convert amplitude into frequency. When the frequency of firing reaches a certain threshold, the brain will suddenly pay attention to it. For this situation Duffiing soft spring ($\alpha<0$) oscillator provides a simple model. Here a frequency threshold causes a sudden jump in amplitude. The forcing term is the input from the sense organ to the brain and the oscillator represents the response of the brain.

(*ii*) Recall, mood and behavioural pattern all suggest catastrophe models. Duffiing oscillator (hard-spring $\alpha>0$ as well as soft spring $\alpha<0$) can explain some of these aspects of brain activity to some degree of our satisfaction.

5. Differential equation for heatbeat and nerve impulse

The simple mathematical models to explain the dynamics of heartbeat and nerve developed by Zeeman are generalizations of the Van der Pol and Lienard equations (Zeeman 1973).

The diastole (relaxed state) is the stable equilibruim state of the heart. The cortraction of heart into systole is a global electrochemical wave emanating from a pacemaker. This wave triggers off rapid contraction of individual muscle fibre which then rapidly relaxes again. The fibre thus obeys the jump return to equilibrium state. This can be described by a system of ordinary differential equations:

$$\epsilon\ \dot{x}_1 = -(x_1^3 - x_1) - C_1 \qquad (5.1)$$

$$\text{and}\quad \dot{C}_1 = x_1 - x_0 \qquad (5.2)$$

where, x_0 is a constant greater than $\sqrt{3}/3$ and ϵ is a small positive constant.

The Eqns. (5.1) and (5.2) can be considered as a model for the heartbeat, x_1 being the length of the muscle fibre and C_1 being some form of electrochemical control which can be measured in several ways.

In a more general manner the fast equation can be chosen of the form :

$$\epsilon\ \dot{x}_1 = -f(x_1, C_1, C_2) \qquad (5.3)$$

To satisfy the smooth return to equilibrium dynamic quality displayed by heart, the simplest form for f can be taken as

$$f = x_1^3 + C_1x_1 + C_2 \qquad (5.4)$$

The catastrophe manifold M is given by :

$$x_1^3 + C_1x_1 + C_2 = 0 \qquad (5.5)$$

The fold curves in M are given by $C_1 = -3x_1^2$. Thus, we have seen earlier that the fold curves when projected into C gives

$$4C_1^3 + 27C_2^2 = 0 \qquad (5.6)$$

This forms a cusp. Outside the cusp M is single sheeted and inside it M is 3-sheeted (Fig.4) M is smooth at O and the cusp exists only in C. The upper and lower sheets are attractors and the middle sheet is a repellor. This lowest degree surface possesses all the required properties. On the basis of Thom catastrophe theory this surface is also unique. We are thus justified both mathematically and biologically in choosing the canonical fast equation as :

$$\epsilon\ \dot{x}_1 = -(x_1^3 + C_1x_1 + C_2) \qquad (5.7)$$

Now we have to consider the slow equations. The cusp catastrophe has no slow equations normally. The control variables C_1 and C_2 due to slow equations possess a dynamic role. Now if x_1 is present in the slow equations, then it can be regarded as a form of feed back on the cusp catastrophe. Based on Hodgkin-Huxley data Zeeman has choosen the slow equations :

$$\dot{C}_1 = -2C_1 - 2x_1 \quad \text{and} \quad \dot{C}_2 = -C_1 - 1$$

There exists a dynamical system on R^3 possessing dynamic qualities displayed by heart muscle fibres and nerve axons. This system can be written as

$$\begin{aligned} \epsilon\ \dot{x}_1 &= -(x_1^3 + C_1x_1 + C_2) \\ \dot{C}_1 &= -2C_1 - 2x_1 \\ \dot{C}_2 &= -C_1 - 1 \end{aligned} \qquad (5.8)$$

The equilibrium condition is given by $\dot{x}_1 = \dot{C}_1 = \dot{C}_2 = 0$. This gives $C_1 = -1$, $x_1 = 1$ and $C_2 = 0$ at equilibrium state.

The Eqn. (5.8) can be applied to the nerve impulse by identifying dynamical variables x_1, C_1, C_2 applicable to this situation. Here C_2 can be considered as membrane potential, $-x_1$ to be correlated with sodium conductance (action) and C_1 to be identified with potassium conductance which begins to change after the action has finished and then it rises. It finally falls smoothly during the smooth return (Fig. 5).

6. Conclusion

In conclusion it will be useful to point out the controversy which has been raised regarding the catastrophe theory and its variety of applications. Some argue that catastrophe theory does not provide us with any new information about the state of a dynamic system. It uses obscure terminology to arrive at obvious conclusions (Zahler & Sussman 1977 a). But there is also a school of thought according to which catastrophe provides us new opportunities to investigate effectively complicated dynamical systems having large number of state and control variables (Zahler & Sussman 1977b).

Biological, economic and social processes involve large number of variables and here catastrophe theory may prove to be quite useful, Kilmister (1979) in a recent review of Zeeman's work asserts that his work does provide a very enlightening lead-in to many applications. It is true that the application of catastrophe theory in physics has caused considerable disappointment amongst those who had unrealistic expectations. Thus it is getting less fashionable. According to the reviewer it is a pity since one gets out of it as much as one puts in.

It is observed that highly optimum systems are quite sensitive to small changes in their control variables and thus resulting into a catastrophic collapse of the system (Thompson & Hunt 1974). Unmindful optimization of a system may prove quite dangerous at some future point after its design and use. Catastrophe theory is a pointer to caution us and so should be used carefully to study a dynamical system.

Particularly in our age there are several ecololgical problems such as pollution of air and water. These can become catastrophic at certain point in future for mankind. The rate at which the atmosphere and rivers, are being polluted, is increasing exponentially with time. At certain stage in future, it may become more than the rate at which the atmosphere and rivers can be kept clean by the natural processes and so a catastrophic situation may arise resulting into destruction of mankind and other living organisms. Our technology at the moment is too much dependent on oil. It is well known that within few decades the world oil sources may get exhausted at this ever-increasing rate of consumptions. This may result into a grinding halt of most of the industries of the world and thereby unleash unprecendented amount of discontentment and violence in human society. Possibly here the solar energy if harnessed properly can help. The another problem is then of deforestation.

Therefore, it is necessary to investigate all those processes which are likely to change catastrophically by the use of new mathematical tool, *viz.*, catastrophe theory. Catastrophe theory which has now become a part of calculus helps us, to anaylse the critical state of various dynamical systems.

It is well known that the development of calculus by Newton and Leibniz, few hundred years ago, has given tremendous boost to our understanding of dynamical systems. No doubt the seed of calculus was sown when Zeno stated his famous paradoxes. But it took approximately 2000 years to bring his ideas to fruition, in the form of calculus. Due to peculiarity of our age we should pay full attention to catastrophe theory to analyse, the dynamical systems as the reaction time for us is much less as compared to our forefathers. Catastrophe theory, I am sure will give us the opportunity to peep into our scientific and technological activity more closely and thus enable us to avoid such situations which may prove dangerous to mankind in near future.

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