Mausam, (1985), **36**, 3, 319-322

517.392

Some numerical methods for resolution of non-linear integral equations*

Y. CHERRUAULT

University of Paris, Paris

सार — हम बहुत सी ऐसी विधियां सुझा रहे हैं जो अरेखीय पूर्णांकीय समीकरण के वियोजन को रेखीय पूर्णांकीय समीकरणों की श्रृंखलों अथवा रेखीय बीजगणितीय प्रणाली में परिणत कर देती है। इन सभी विधियों की आंकिक जांच कर ली गई है।

ABSTRACT. We suggest several methods which bring back the resolution of non-linear integral equations to a succession of linear integral equations or to a linear algebraic system. All these methods have been tested numerically.

1. Introduction

We will examine integral equations such that

$$u(x)=\lambda \, . \int_a^b K(x,t) \, . \, g[u(t)] \, dt+f(x) \qquad (1)$$

where, functions K, g, f are given.

We must calculate $u(x)$. We are concerned by the search of *simple* numerical methods for the determination of $u(x)$.

We will suppose that Eqn. (1) admits a unique real solution for f, K, g, given. Theorems of existence and unicity are given by Miklin and Tricomi. In Anselone, Cherruault and Glowinski's publications we find numerical methods which bring through the resolution of non-linear algebraic systems. In the best case, R. Glowinski obtains a succession of non-linear algebraic equations. We will give here simpler methods, easy to use.

2. First method

Reduction to a succession of linear integral equations (Cherruault *et* Guillez) :

Starting from Eqn. (1) we build $u_o(x)$ solution of

$$u_o(x)=\lambda \, . \int_a^b K_o(x,t) \, . \, g[u_o(t)] \, dt+f(x) \qquad (2)$$

where, $K_o(x,t)$ is an approximation of $K(x,t)$ which may be crude.

Let us set $R(x,t)=K(x,t)-K_o(x,t)$ and

$$n(x)=\lambda \, . \int_a^b R(x,t) \, . \, g[u_o(t)] \, dt.$$

It is easy to show that $v(x)=u(x)-u_o(x)$ is solution of

$$v(x)=n(x)+\lambda \, . \int_a^b K(x,t) \, . \, [g(u)-g(u_o)] \, dt \qquad (3)$$

We can notice that $n(x)$ is known if u_o is calculated. Setting $g(u)-g(u_o)=(u-u_o) \, . \, G(u_o,u)=G(u_o,u) \, . \, v(t)$ we obtain the integral equation

$$v(x)=n(x)+\lambda \, . \int_a^b K(x,t) \, . \, G(u_o,u) \, . \, v(t) dt \qquad (4)$$

The first method will give the result of an interesting approximation of $G(u_o,u)$.

Define the numerical algorithm as below :

Setting $u_{n+1}=u_o+v_n$

$u_1(x)=u_o+v_o$ is obtained from

$$v_o(x)=n(x)+\lambda . \int_a^b K(x,t) . \, G(u_o,u_o) \, . \, v_o(t) dt$$

which is a linear integral equation.

Generally $v_n(x)$ is given by resolution of the *linear* integral equation :

$$v_n(x)=n(x)+\lambda . \int_a^b K(x,t) \, . \, G(u_o,u_n) \, . \, v_n(t) \, dt \qquad (5)$$

There exist a lot of simple numerical methods to solve the Eqn. (5).

*Paper was presented in the symposium "Indo-French School on recent advances in Computer Techniques in Meteorology, Biomechanics and Applied Systems" held at I. I. T., New Delhi, 4-13 February 1980

Practically, as we will see later, this method converges quickly (in three of four iterations). Convergence may be improved by using "good" approximations for $G(u_o, u)$.

The following can be established (Cherruault *et* Guillez)

Theorem 2.1—If we suppose $| G(u_o, u) | \leqslant G < \infty$), $| K(x,t) | \leqslant k < \infty$ and $n(x) \in H^1(a,b)$, then v_n solution of Eqn. (5) converges towards $u(x)$ solution of Eqn. (1).

For the proof we use bounded sequences and weak compacticity.

It is possible to apply this method to more general integral equations.

For instance we can deal with :

(a) $u(x)=f(x)+\int_a^b K(x,t) \, . \, g(u,t) \, dt$

(b) $u(x)=f(x)+\lambda . \int_a^b K[x, t, u(t)] \, dt$

3. A second method

This method leads to the resolution of a sequence of *linear* systems. It can be considered as a discretization of the previous method.

Let us consider the equation

$$u(x)+\int_0^1 K(x,t).u^3(t) \, dt=f(x) \qquad (6)$$

We can, of course, approximate Eqn. (6) by :

$$u(x)+\sum_{j=0}^{N-1}\int_{jh}^{(j+1)h} K(x,t).u^3(t) \, dt=f(x) \qquad (7)$$

which gives for $x=ih$

$$u(ih)+\sum_{j=0}^{N-1}\int_{jh}^{(j+1)h} K(ih,t) \, . \, u^3(t) \, dt=f(ih)$$

$$Nh=1 \; ; \; i=0,\ldots\ldots, N \qquad (8)$$

On $[jh,(j+1)h]$ let us approach $u^3(t)$ by

$u^3(jh)+3u^2(jh).[u((j+1)h)—u(jh)]$

It's a linearization near $t=jh$. It's valid because $u^3(t)$ is used on an interval of length h with $h \to 0$.

Remark 1—If we had $g(u)$ instead of $u^3(t)$, we could approximate $g(u)$ by $g(u_o)+(u-u_o).g'(u_o)$.

Let us now build the iterative diagarm.

$$u_n(ih)+\sum_{j=0}^{N-1} h \, . \, K[ih,(j+1)h] \, . \, \Big\{u^3_{n-1}(jh)+$$

$$+3u^2_{n-1}(jh)[u_n((j+1)h)-u_{n-1}(jh)]\Big\}=f(ih)$$

$$i=0,\ldots,N \; ; j=0,\ldots,N-1, \; n=1, \; 2,\ldots \qquad (9)$$

where $u_o(jh)$ have been calculated before or are arbitrarily fixed.

For a fixed n Eqn. (9) is a *linear* system whose $u_n(jh)$ are unknown $(j=0,\ldots,N)$. The system (9) will give an approximate solution or (6) or (7) by successive resolution of linear systems.

For convergence proof, we can use the same method as in § 2. Nevertheless it will be necessary to take into account the discretization.

Remark 2—The scheme (9) is not unique. More generally, we can choose more sophisticated approximations of

$$\int_{jh}^{(j+1)h} \text{ and of } u^3(t) \text{ or } g(u).$$

Remark 3— Practically it will be useful to start with well adapted $u_o(jh)$. A. Caron (Lab. Medimat) (*see* Ref.) has noticed that a good approximation was given by :

$$u_o(ih)=\frac{f(ih)}{1+(e^{ih}-1)/ih} \qquad i=1,\ldots,N$$

$$u_o(0)=\frac{f(0)}{2} \qquad (10)$$

with the kernel $K(x,t)=e^{xt}$. In general case we use

$$u(x)=\frac{f(x)}{1+\int_0^1 K(x,t) \, dt} \qquad (11)$$

Morever it's possible to suggest a method involving resolution of a *sequence* of *linear* equations instead of the previous linear system. In fact let's associate to Eqn. (8) the iterative scheme.

$$u_n(ih)+\sum_{j=0}^{i-1}\int_{jh}^{(j+1)h} K(ih,t) \, . \, u_n^3(t) \, dt+$$

$$+\sum_{j=i}^{N-1}\int_{jh}^{(j+1)h} K(ih,t) \, . \, u^3_{n-1}(t) \, dt=f(ih)$$

$$i=1,\ldots,N-1 \qquad (12)$$

Then, we have

$$\int_{jh}^{(j+1)h} u_n^3(t) \,.\, K(ih, t)\, dt$$

$$\# h \,.\, K(ih, (j+1)h)\,[u^3_{n-1}(jh)+3u^2_{n-1}(jh) \,.$$
$$.\,[u_n((j+1)h)-u_{n-1}(jh)]]$$

so that (12) becomes

$$u_n(ih)+\sum_{j=0}^{i-1} h \,.\, K[ih, (j+1)h].$$

$$.\,[u^3_{n-1}(jh)+3u^2_{n-1}(jh) \,.\, [u_n((j+1)h)-u_{n-1}(jh)]]+$$

$$+\sum_{j=i}^{N-1}\int_{jh}^{(j+1)h} K(ih, t) \,.\, u^3_{n-1}(t)\, dt=f(ih)$$

$$i=1, \ldots, N. \qquad (13)$$

where $u_o(jh)$ are fixed or calculated as in remark 3.

Eqn. (13) shows us that we obtain $u_n(h)$, $u_n(2h)$, ... by resolution of a sequence of *linear* equations.

Remark 4—As for previous method it is possible to improve the approximation of

$$\int_{jh}^{(j+1)h} K(ih, t) \,.\, u_n^3(t)\, dt\,.$$

Moreover Eqn. (13) is not complete without $u_n(0)$. We have taken

$$u_n(x)=-\sum_{j=0}^{N-1}\int_{jh}^{(j+1)h} K(ih, t)\,.\,u^3_{n-1}(t)\, dt+f(0)\,.$$

4. Numerical results*

For the method described in § 2, numerical trials have been done on a *pocket* calculator.

The following equations have been solved

$$.\; u(x)=x+\int_{-1}^{1} e^{xt} \,.\, u^3(t)\, dt$$

We have choosen $K_o=1+xt$, $R=e^{xt}-1-xt$

$$.\; u(x)=f(x)-\int^{1} e^{xt} \,.\, u^3(t)\, dt$$

with $f(x)=\dfrac{e^x-1}{x}+1$ which implies $u(x)\equiv 1$

We have taken $K_o\equiv 1$ which is a crude approximation of e^{xt}. Errors (in L^2 norm) are between 1.10^{-3} and 1.10^{-4}.

With the methods described in § 3 we have solved (*see* Cherruault *et* Guillez and Caron) :

$$.\; u(x)=f(x)-\int_0^1 e^{xt} \,.\, u^3(t)\, dt$$

with

$$f(x)=1+\frac{e^x-1}{x}\quad [u(x)\equiv 1]$$

then with

$$f(x)=x+\frac{e^x}{x}-\frac{3e^x}{x^2}+\frac{6e^x}{x^3}-6\,\frac{(e^x-1)}{x^4}$$

and

$$f(x)=x^2+\frac{e^x}{x}-\frac{6e^x}{x^2}+\frac{30e^x}{x^3}-\frac{120e^x}{x^4}+\frac{360e^x}{x^5}$$
$$-\frac{720e^x}{x^6}+720\,\frac{(e^x-1)}{x^7}\,.\;[u(x)=x^2]$$

Precision is similar to the previous method. It is possible to improve precision by using best quadrature formula (Gauss).

5. Conclusions

The described methods are simple and easy to use because they lead to linear equations or systems. Moreover they can be extended, without difficulties, to integral equations or systems of several variables. We can also apply these methods to singular, non-linear integral equations.

References

Anselone, *Non-linear integral equations*, Wisconsin.

Caron, A., Résolution numérique d'équations intégralse. Thése de 3e cycle (*Paris* 6)—à paraitre.

Cherruault, Y., Approximation d'opérateurs linéaires et applications. Dunod.

Cherruault, Y. *et* coll.—Résolution d'équations par des méthodes d'optimisation. Tomes I et II. Contrat D.G.R.S.T 68.01. 452.

*The methods of § 3 have been numerically tested by A. Caron (Medimat), (*see* Ref.)

Cherruault, Y. *et* Guillez., A., Méthodes numériques pour la résolution d'équations intégrales non linéaires. Revue du Cethedec. (à paraitre).

Glowinski, R., Etude et approximation de quelques problems intégraux *et* integro-differentiels. These d'Etat, Paris, 1970.

Miklin, S., *Integral equations*, Pergamon Press.

Pogorzelski, W., *Integral equations and their applications*, Pergamon Press.

Tricomi, F.G., *Integral equations*, Interscience.