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Galerkin B-spline technique for nonlinear parabolic problems*

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सार — अरेखीय पैराबोलीय आंशिक अवकल समीकरण के हल के लिए एक अभिकलनीय योजना बनाई गई है। प्रस्तावित योजना जो समय की दृष्टि से द्वितीय कोटि की यथार्थ है। एक संयुक्त क्वासीलीनिअराइजेशन तथा गैलर्किन विधि जिसमें घन बी-स्प्लाइन वाला आधार इस्तेमाल किया गया है, पर आधारित है। योजना के परीक्षण/जांच के लिये एक निर्देशात्मक उदाहरण को हल किया गया है। उपलब्ध परिणामों से आंकिक परिणाम अच्छा मेल खाते हैं।

ABSTRACT. A computational scheme for the solution of nonlinear parabolic partial differential equations is developed. The proposed scheme, which is second order accurate in time, is based upon a combined approach of quasi-linearization and the Galerkin method wherein a basis consisting of the cubic B-splines has been used. An illustrative example is solved to test the scheme. The numerical results are in good agreement with the available results.

1. Introduction

The prediction of meteorological parameters by the solution of hydrodynamic equations governing atmospheric motions is known as numerical weather prediction. Numerical weather prediction involves, more often than not, the solution of nonlinear parabolic partial differential equations. Since the analytic solutions for these equations are almost never available a resort to the numerical methods seems to be a must. Of the numerical methods, the difficulties associated with the finite difference methods are well known. The finite element methods, although somewhat more difficult to program for a computer than the finite difference methods, have certain inherent advantages. Among the finite element methods, the Galerkin methods possess a very desirable feature, namely, the form of the solution. In most of the cases, the solution is a smooth function which is a piecewise polynomial. This has led several research workers to the development of algorithms, for the solution of nonlinear parabolic equations, which make use of the Galerkin method. The most prominent among them is Douglas and Dupont (1970, 1973). It is known that the Galerkin method with a basis of B-splines yields results of higher order accuracy. Davies (1970) has obtained the numerical solutions of the primitive equations in one dimension using the Galerkin method with a basis of B-splines, for a number of boundary conditions. He has also obtained a solution of the nonlinear Burgers' equation (1978). Murphy (1975) has presented cubic spline Galerkins approximations to parabolic systems with coupled nonlinear boundary conditions. In this paper, a scheme is proposed for solving nonlinear parabolic equations. The proposed scheme is based upon a combined approach of quasilinearization (Bellman and Kalaba 1965) and the Galerkin method wherein a basis consisting of the cubic B-splines has been used.

2. Formulation of the problem

Consider the nonlinear parabolic equation:

$$u_t - u_{xx} = F(x, t, u, u_x),\ x \in D,\quad t > 0 \tag{1}$$

with the initial condition,

$$u(x, 0) = u_o(x),\ x \in D \tag{2}$$

and the boundary conditions

$$u(x, t) = g(x, t),\quad x \in \partial D,\quad t \geqslant 0 \tag{3}$$

where, for simplicity, let us assume that

$$D = [0, a],\quad a \in \mathbf{R},\ a > 0.$$

We quasilinearize Eqn. (1) to obtain the following sequence of linear partial differential equations :

$$u_t^{n+1} - u_{xx}^{n+1} = F\left(x, t, u^n, u_x^n\right) + \left(u^{n+1} - u^n\right) \times$$

$$\times F_n\left(x, t, u^n, u_x^n\right) + \left(u_x^{n+1} - u_x^n\right) F_{u_x}\left(x, t, u^n, u_x^n\right),$$

$$x \in D,\ t > 0,\ n = 0, 1, 2, 3, \ldots, \tag{4}$$

where a subscript denotes partial differentiation and u^{n+1} denotes the $(n+1)$ st member of the sequence $\{u^k\}k \geqslant 0$. The initial and boundary conditions corresponding to Eqns. (2) and (3) are :

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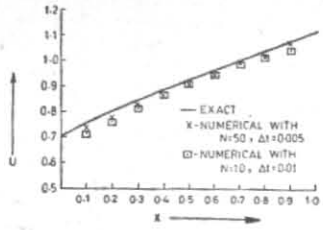

Fig. 1

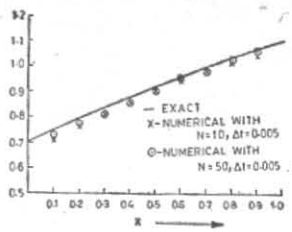

Fig. 2

Figs. 1 & 2. Solution of test example at $t=0.02$

$$u^{n+1}(x, 0)=u_0(x),\ x \in D \tag{5}$$

and

$$u^{n+1}(x, t)=g(x, t),\ x \in \partial D,\ t \leqslant 0 \tag{6}$$

respectively. Eqn. (4) can be written in the form :

$$u_t^{n+1}-u_{xx}^{n+1}=f\left(x, t, u^{n}, u_x^{n}, u^{n+1}, u_x^{n+1}\right)$$

$$x \in D,\ t>0,\ n=0, 1, 2, 3, \ldots, \tag{7}$$

Thus Eqns. (1)-(3) have been transformed into Eqns. (7), (5) and (6). We now use the Galerkin method as proposed by Douglas and Dupont (1970). The weak form of Eqn. (7) is

$$<x_t^{n+1}, v>+<u_x^{n+1}, v_x>=<f(u^{n}, u^{n+1}), v>,$$

$$v \in H_o(D),\ t>0,\ n=0, 1, 2, 3, \ldots, \tag{8}$$

where,

$$<p, q>=\int_D p(x)\, q(x)\, dx$$

$$f(u^{n}, u^{n+1})\equiv f(x, t, u^{n}, u_x^{n}, u^{n+1}, u_x^{n+1})$$

and $H_o^1(D)$ is the closure of $C_o^\infty(D)$, the set of infinitely differentiable functions with compact support in D with respect to the norm defined by :

$$\left|\left| u \right|\right|^2_{H_o^1(D)}=\left|\left| u \right|\right|^2_{L^2(D)}+\left|\left| \frac{\partial u}{\partial x} \right|\right|^2_{L^2(D)}$$

The initial condition corresponding to Eqn. (5) is given by :

$$<u^{n+1}(.,0), v>=<u_0, v>,$$

$$v \in H_o^1(D),\ n=0, 1, 2, 3, \ldots. \tag{9}$$

We approximate u^{n+1} by approximating $w^{n+1}=u^{n+1}-g$. Note that w^{n+1} satisfies:

$$<w_t^{n+1}, v>+<w_x^{n+1}+g_x, v_x>=<\tilde{f}(w^n, w^{n+1}), v>$$

$$v \in H_o^1(D),\ t>0,\quad n=0, 1, 2, 3, \ldots \tag{10}$$

$$<w^{n+1}(x, 0), v>=<w_o, v>,\ v \in H_o^1(D)$$
$$n=0, 1, 2, 3, \ldots \tag{11}$$

where,

$$\tilde{f}(w^n, w^{n+1})=f(x, t, w^n+g, w^n{}_x+g_x, w^{n+1}+g;$$

$$w_x^{n+1}+g_x)-\frac{\partial g}{\partial t}(x, t)$$

$$w_o(x)=u_o(x)-g(x, 0).$$

We now choose a set $\{V_i\}$ of functions from $H_o^1(D)$ which are linearly independent. Let M denotes the subspace of $H_o^1(D)$ spanned by this set. We approximate w^{n+1} by W^{n+1}.

Let

$$W^{n+1}=\sum_i \alpha_i^{n+1}(t)\, V_i(x) \tag{12}$$

where $\alpha_i^{n+1}(t)$ are unknown coefficient functions to be determined such that W^{n+1} satisfies :

$$<W_t^{n+1}, V>+<W_x^{n+1}+g_x, V_x>$$

$$=<\tilde{f}(W^n, W^{n+1}), V>,$$

$$V \in M,\ t>0,\ n=0, 1, 2, 3, \ldots. \tag{13}$$

$$<W^{n+1}(.,0), V>=<w_o, V>,\ V \in M,\ n=0,1,2,3,\ldots \tag{14}$$

We denote by W_j^{n+1} the numerical approximation for W at the jth time level. Thus

$$W_j^{n+1}=\sum_j \alpha_i^{n+1}(t)_j\, V_i(x)=\sum_j \alpha_{i,j}^{n+1} V_i \tag{15}$$

We now use the Crank-Nicolson-Galerkin approximation in Eqn. (13) to obtain:

$$<\frac{W_j^{n+1}-W_{j-1}}{\Delta t}, V>+<\left(\frac{W_j^{n+1}+W_{j-1}}{2}\right)_x+g_x, V_x>$$

$$=<\tilde{f}\left(W_j^{n}, \frac{W_j^{n+1}+W_{j-1}}{2}\right), V>,$$

$$V \in M,\ n=0, 1, 2, 3, \ldots,\ j=1, 2, 3, \ldots. \tag{16}$$

where W_{j-1} denotes the value of W at the $(j-1)$st time level. Eqn. (14) is replaced by :

$$<W_o^{n+1}, V>=<w_0, V> \tag{17}$$

TABLE 1

Values of the analytical and numerical solution at a few mesh points at $t=1.0$ with $g(x, t)=u(x, t)$

x	Analytical solution $u(x, t)$	Numerical solution $U(x, t)$
0.0	1.0986118	1.0986118
0.2	1.1631498	1.1631498
0.4	1.2237749	1.2237749
0.6	1.2809334	1.2809334
0.8	1.3350000	1.3350000
1.0	1.3862934	1.3862934

Eqns . (16 and (17) are basic equations for the solution of the given problem.

3. Description of the basis functions

We have used the cubic B-splines B_i for V_i, the basis functions. A cubic spline function is a piecewise cubic polynomial that is twice continuously differentiable. The cubic B-splines are defined as follows (Prenter 1975) : Divide the interval $[0, a]$ at N equidistant points x_i, $0 \leqslant i \leqslant N$, such that :

$$0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = a$$

Let $x_i - x_{i-1} = h$, $1 \leqslant i \leqslant N$. Introduce four more points x_{-2}, x_{-1}, x_{N+1} and x_{N+2} such that :

$$x_{-2} < x_{-1} < x_0 \text{ and } x_{N+2} > x_{N+1} > x_N$$

Then the cubic B-splines B_i are defined by :

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x-x_{i-2})^3, & \text{if } x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 - \\ \quad -3(x-x_{i-1})^3, & \text{if } x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 - \\ \quad -3(x_{i+1}-x)^3, & \text{if } x \in [x_i, x_{i+1}] \\ (x_{i+2}-x)^3, & \text{if } x \in [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise } i = 0, 1, 2, \ldots N \end{cases}$$

Since the functions B_o, B_1, B_{N-1} and B_N are nonzero at the boundaries, we have used $\{B_2, B_3, \ldots, B_{N-2}\}$ as the set of basis functions.

4. Numerical method

The solution U_j at the jth time level is obtained from Eqns. (16) and (17) by an iterative procedure described below :

Step 1 : Set W_j^o equal to W_{j-1}, the value of W at the $(j-1)$st time level.

Step 2 : Substitute for W_j^1, from Eqn. (15) and for W_{j-1}, W_j^o in Eqn. (16). This yields a linear algebraic equation in α's for each value of $V \in M$.

Step 3 : Assign a set of different values for $V \in M$ to obtain a set of linear algebraic equations in α's.

Step 4 : Solve this set to obtain α's.

Step 5 : Substitute back for α's in Eqn. (15) to obtain W_j^1.

Step 6 : Use W_j^2, obtained above, and W_{j-1} to obtain W_j^2 in the same manner as described above (Step 2 to Step 5).

Step 7 : Repeat the above procedure to obtain W_j^3, W_j^4, ..., until the difference between the values of two successive members of the sequence $\{W_j^k\}_{k \geqslant 0}$ becomes less than a given tolerance E (usually of the order of 10^{-5}). Let n_k be the smallest integer such that

$$| W_j^{n_k+1} - W^{n_k} | < E$$

Then the value W_j of W at the jth time level is obtained by setting W_j equal to W_j^{nk+1}.

Step 8 : Finally,

$$U_j = W_j + g$$

gives the solution U_j at the jth times level.

5. Numerical results

To test the proposed scheme we have solved

$$u_t = u_{xx} + e^{-u} + e^{-2u}, \quad 0 < x < 1, \quad t > 0 \tag{18}$$

together with the initial condition :

$$u(x, 0) = \ln (x+2), \quad 0 < x < 1 \tag{19}$$

and the boundary conditions :

$$u(0, t) = \ln (t+2), \qquad t \geqslant 0 \tag{20}$$

$$u(1, t) = \ln (t+3), \qquad t \geqslant 0 \tag{21}$$

The analytical solution of this problem is given by (Hopkins and Wait 1978) :

$$u(x, t) = \ln (x+t+2) \tag{22}$$

To solve problem (18-21), we have considered two different boundary functions g. In the first instance, we have chosen $g(x, t) = g_1(x, t) = u(x,t)$. The numerical solution obtained in this case is virtually in distinguishable from the analytical solution on any reasonable scale. Table 1 presents the values of the numerical solution and the analytical solution at a few mesh points at $t=1.0$.

When $g(x, t)$ was changed to $g_2(x, t) = \ln (x^2+t+2)$, it was observed that the accuracy of the results is slightly hampered. Figs. 1 and 2 show the analytical and numerical solutions at $t=0.02$ for different values of

N, where $(N-3)$ is the number of basis functions and $\triangle t$, the time step.

It is found that the proposed scheme is fast and accurate. It can be used for solving other nonlinear parabolic equations.

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