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Kinematics of the transport of momentum by tilted troughs

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ABSTRACT. It is shown that in x, y plane, while the direction of tangent to a tilted trough indicates whether the u-momentum flux is northwards or southwards, the curving or the concavity of the tilted trough indicates whether there is convergence or divergence of the momentum flux. Concavity of a trough towards east indicates divergence of flux and concavity towards west indicates convergence of mementum flux. The magnitude of this divergence for parabolically tilted trough is given by average value of (meridional velocity)^2/(semi latus rectum of the parabola). Formulae are also given for momentum flux and its divergence when the tilted trough is a cubic curve. Qualitative pattern of the momentum flux and its divergence are given for various trough patterns to enable an analyst to infer the same from a mere look at the pattern on a chart.

1. Introduction

Dynamical importance of tilted troughs was first recognised by Charney (1947) in the vertical plane and by Starr (1948) and Kuo (1949) in the horizontal plane. Permanent shape tilted troughs consistent with vorticity equation were studied by Machta (1949) and Arakawa (1953). Several empirical and theoretical studies have since shown the importance of these tilts in exchanges of momentum and energy in the horizontal and in the vertical directions.

It is known that in a horizontal plane, NE-SW tilt of troughs and ridges indicates northward flux of westerly momentum. The present author is not aware of equally simple quantitative rule for estimating the divergence or convergence of these momentum fluxes or their variations in y-direction by mere look at the tilt. The purpose of the present note is to give simple rules for inferring such divergence or convergence of momentum flux by mere look at the tilt.

2. Tilted trough as a straight line

Let the stream function at an instant of time be given by

$$\psi = -\alpha(y) + A \sin [k(qy - x + r)] \qquad (2\cdot1)$$

where $\alpha(y)$ represents any continuous and differentiable function of y. Then $\partial\alpha/\partial y$ represents a zonal current capable of several physically reasonable variations in y-direction. A is a constant denoting amplitude of the sinusoidal perturbation superimposed on the zonal current $\partial\alpha/\partial y$; k is the wave number in x-direction, q and r are constants. Equi-phase lines of the sinusoidal perturbation are straight lines given by

$$qy - x + r = \text{constant} \qquad (2\cdot2)$$

$$\text{slope of trough} = 1/q \qquad (2\cdot3)$$

We also have,

$$\left.\begin{aligned} u &= -\frac{\partial\psi}{\partial y} = \frac{\partial\alpha}{\partial y} - Akq\cos[k(qy - x + r)] \\ v &= \frac{\partial\psi}{\partial x} = -Ak\cos[k(qy - x + r)] \end{aligned}\right\} \qquad (2\cdot4)$$

$$\left.\begin{aligned} \bar{u} &= \frac{\partial\alpha}{\partial y}\ ;\ u' = -Akq\cos[k(qy - x + r)] \\ \bar{v} &= 0\ ;\ v' = -Ak\cos[k(qy - x + r)] \end{aligned}\right\} \qquad (2.5)$$

$$\overline{u'v'} = \tfrac{1}{2} A^2k^2q \qquad (2.6)$$

$$\frac{\partial}{\partial y}\,\overline{(u'v')} = 0 \qquad (2.7)$$

If q is positive, the troughs and ridges slope in NE-SW direction and $\overline{u'v'}$ is positive. If q is negative, the reverse holds. In either case, divergence of momentum flux is zero. This is a well-known result.

3. Tilted trough as a parabolic curve

Let the stream function be given by

$$\psi = -\alpha(y) + A \sin [k\{(p/2)y^2 + qy - x + r\}] \quad (3\cdot1)$$

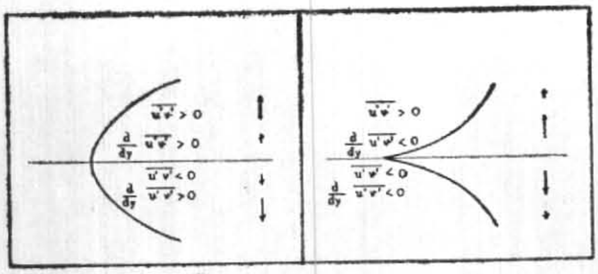

Fig. 1 (a) Fig. 1 (b)

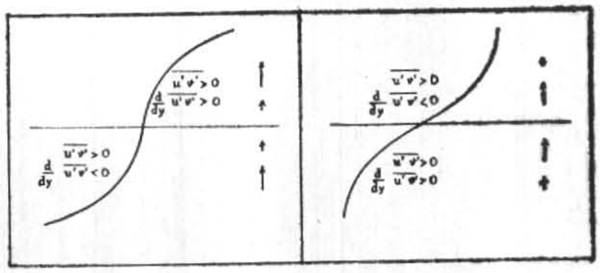

Fig. 2 (a) Fig. 2 (b)

Then equi-phase curves are given by

$$(p/2)\, y^2 + qy - x + r = \text{constant} \tag{3.2}$$

$$\text{Slope of trough} = 1/(py+q) \tag{3.3}$$

$$\left.\begin{aligned} \overline{u'v'} &= \tfrac{1}{2} A^2k^2 (py+q) \\ \frac{\partial}{\partial y}\,(\overline{u'v'}) &= \tfrac{1}{2}A^2k^2 p \end{aligned}\right\} \tag{3.4}$$

The equi-phase curves are a system of parabolas with axis in x-direction and length of latus rectum $=2/p$. If p is positive, the curves are concave to the east; if p is negative, the curves are concave to the west.

Since the sign of $(\partial/\partial y)\,(\overline{u'v'})$ in equation (3.4) is determined by the concavity p, it follows that parabolic troughs with concave side towards east are associated with divergence of momentum flux; those with concave side towards west are associated with convergence of momentum flux. From equations (3.4) it also follows that the magnitude of flux divergence is given by

$$\left|\frac{\partial}{\partial y}\overline{(u'v')}\right| = \frac{(\text{amplitude of perturbation})^2\ (\text{wave number in } x)^2}{\text{Latus rectum of parabola}}$$

$$= \frac{(\text{maximum value of } v')^2}{\text{Latus rectum of parabola}}$$

$$= \frac{\text{Average value of } v'^2}{\text{semi-latus rectum of parabola}} \tag{3.5}$$

The sense of divergence of momentum flux could be anticipated from qualitative analysis presented by Yeh (1951) and Starr (1966, p. 20). Here, our treatment is quantitative.

4. Tilted trough as a cubic curve in y

So far, the divergence of flux has been either zero (straight line troughs) or constant (parabolic troughs). However, for the same wave number k in x-direction, we can make divergence of flux also vary in y-direction by taking

$$\psi = -\alpha\,(y) + A \sin\left\{ k\left(\frac{n}{6}\,y^3 + \frac{p}{2}y^2 + qy - x + r\right)\right\}$$

It is then easy to see that (4.1)

$$\overline{u'v'} = \tfrac{1}{2}\, A^2k^2 \left(\frac{n}{2}\, y^2 + py + q\right) \tag{4.2}$$

$$\frac{\partial}{\partial y}\,(\overline{u'v'}) = \tfrac{1}{2}\, A^2k^2\,(ny+p) \tag{4.3}$$

$$\frac{\partial^2}{\partial y^2}(\overline{u'v'}) = \tfrac{1}{2}\, A^2k^2 n \tag{4.4}$$

The equi-phase curves are then given by the cubic curves

$$\frac{n}{6}y^3 + \frac{p}{2}\, y^2 + qy - x + r = \text{constant} \tag{4.5}$$

Based on the above analysis, we give sketches of some tilted troughs (Figs. 1 to 4) indicating the sign of $\overline{u'v'}$ and of $(\partial/\partial y)\,\overline{(u'v')}$ The variations of $(\partial/\partial y)\ \overline{(u'v')}$ in y-direction are also indicated. In each figure, continuous curve represents the orientation of a trough line. Arrows on the right hand side of each figure depict the sense of momentum flux, length of the arrowed line representing schematically the magnitude of this momentum flux. Differences in the lengths of the arrowed lines immediately indicate whether there is convergence or divergence of the momentum flux. For example, in Fig. 1(a) the momentum flux is away from the east-west axis of the parabolic trough line with divergence of flux throughout the region. In Fig. 1(b), the sense of momentum flux is the same as in Fig. 1(a) but there is convergence of flux throughout the region. In the eight configurations shown in Figs. 1 to 4, there is a cusp only in Figs. 1(b) and 3(b). In both

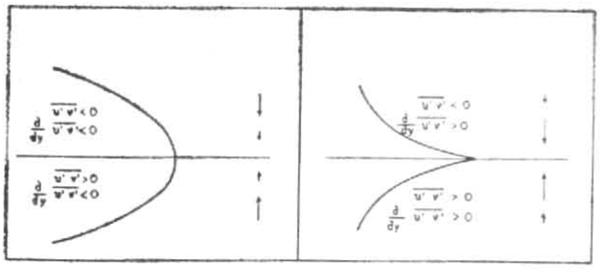

Fig. 3(a) Fig. 3(b)

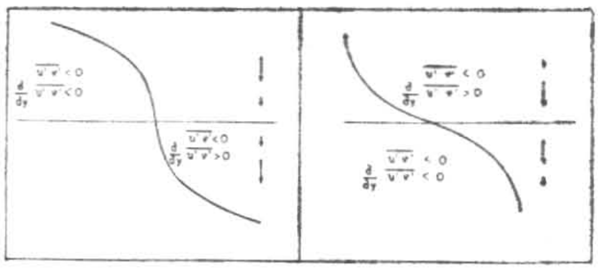

Fig. 4(a) Fig. 4(b)

these configurations, there is discontinuity in the flux at the east-west axis of the trough line. Such discontinuities are generally not seen on the synoptic charts. Continuous variations are frequently observed.

5. Extension of results to x, z plane

What has been said of x, y plane above holds for x, z plane as well; in that case, y in the above formulations has to be replaced by z and v by w.

6. Limitations of the study

This study has the following limitations :

(*i*) The study is kinematic rather than dynamic in as much as it does not deal with dynamic forces causing or maintaining the tilts and the transports associated with them.

(*ii*) Time variations of the basic flow patterns have not been brought under discussion. Hence, the study is diagnostic rather than prognostic.

(*iii*) Perturbation has been introduced in the form of a single wave component of which the amplitude does not vary in north-south direction. Real perturbations observed in nature are far more complicated.

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