Mausam, (1989), **40**, 2, 131-136

551.509.313 : 551.513

Topographic waves in barotropic flow

ANIL KUMAR JAIN

Meteorological Office, Agartala Airport, Tripura

(*Received 3 March 1988*)

सार — अश्यान दाब घनत्वी बीटा प्लेन निदर्श में स्थलाकृतिक वायु प्रेरित तरंग से युक्त आधारभूत अवस्था की अस्थिरता पर रैखिक और अरैखिक प्रभावों के अध्ययन के लिए फूरिये श्रेणी विश्लेषण की तकनीक का प्रयोग किया गया। इसमें यह दर्शाया गया है कि अस्थिरता संबंधी एक विश्लेषणात्मक अध्ययन संभव है यदि मूलबिंदु के संबंध में क्षोभ के समान तरंग के आयाम का फलन अंतरिक्ष में या तो सम हो या विषम हो। इस शोधपत्र में यह अध्ययन पहले वाले मामले के संबंध में किया गया है।

ABSTRACT. Technique of Fourier series analysis is used to study linear and non-linear effects on the instability of a basic state consisting of a topographically forced wave in an inviscid barotropic beta plane model. It is shown that an analytical study on instability is possible if amplitude of wave like perturbation is either an even or odd function in space about origin. The study is carried out in the present paper for former case.

1. Introduction

In a series of recent papers (Charney and Devore 1979, Charney and Strauss 1980, Hart 1979, Deininger 1981, Pedlosky 1981), concept of topographic instability and theory on resonant topographic waves in baroclinic and barotropic flows are developed. Holopainen (1978) has shown importance and significant relative contribution of transient eddies in maintenance of horizontal flux of relative vorticity. Also pointed out by Pedlosky (1981), the models in the papers cited above suffer from either severe truncations or assumptions sometimes unrealistic. It can be argued that truncations or assumptions are not adopted to simplify analysis of basic vorticity equation by all the investigators. Imposition of a constraint is found unavoidable for want of an additional equation or condition to conclude results from the analysis. The governing equation cannot impose restriction on the field of perturbations that may exist in the atmosphere.

The atmosphere is baroclinic. However, the behaviour of a barotropic atmosphere, surely has a mapping in the baroclinic atmosphere. Hence, a result that is obtained for barotropic model, should be treated as the result that provides basic understanding of the physics involved.

Deininger (1981) used theory of perturbation with infinite series solution of differential equation to study interaction between standing and transient eddies in the barotropic flow in the atmosphere. He proposed truncation to make analysis possible. Although it is possible to carry out analysis for less restrictive truncation yet truncation is a must for an analysis. The analysis would be subjected to criticism for any choice of finite truncation. In the present paper similar theory is used for linear and non-linear analyses. However the analysis, presented here, is more rigorous. It is believed in present work that interaction between standing and transient eddies is function of topography and perturbation. For all type of perturbations that may exist in the atmosphere, topography need not to result instability. However, there may exist some perturbations that exhibit instability due to interaction with topography. Hence, a classification of perturbations is likely to be possible such that perturbations from a classified group may result in an instability. Classifying perturbations in three groups, *i.e.*, wave like perturbations propagating in (k_0, l_0) direction with frequency λ and either an even or odd or mixed amplitude in space about origin, a linear and non-linear analysis is carried out for first classification in the present work.

2. The model

The quasi-geostrophic vorticity equation for a homogeneous, barotropic fluid on the β-plane can be written in non-dimensional form as (Pedlosky 1979) :

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2\Psi+\beta y+h)=0 \qquad (2.1)$$

where Ψ is the geostrophic stream function whose x and y derivatives give v and $-u$ respectively. The planetary vorticity gradient is β.

If h_B is height of the topography then :

$$h = h_B/\epsilon D \qquad (2.2)$$

where,

$$\epsilon = U / f_0 L \tag{2.3}$$

is the Rossby number and D is the mean depth of the fluid, U the characteristic horizontal velocity and L is a characteristic horizontal length. As usual, f_0 represents Coriolis parameter. If topography is assumed to be sinusoidal, *i.e.*,

$$h = h_0 \sin \theta \tag{2.4a}$$

where,

$$\theta = kx + ly \tag{2.4b}$$

We obtain an equation whose stability is studied by Charney and Flierl (1981) and Deininger (1981). The exact solution of this equation is :

$$\Psi = -U\,y + F \sin \theta \tag{2.5}$$

where,

$$F = h_0 U / N^2 (U - C) \tag{2.6}$$

and

$$C = \beta / N^2 \quad ; \quad N^2 = k^2 + l^2 \tag{2.6a}$$

Deininger (1981) called solution, Eqn. (2.5), as barotropic topographically forced wave. He used assumption, "The barotropic topographically forced wave will represent the vertically averaged standing wave in the atmosphere and the travelling linear, barotropic disturbance which develops on the topographic wave will represent the large scale vertically averaged transient atmospheric disturbances" to study interaction between standing and transient eddies. Proceeding on same lines, we would formulate the non-linear problem.

3. Formulation of non-linear problem

Let Φ be the disturbance stream function. Then solution of Eqn. (2.1) may be written as :

$$\Psi = -\,U\,y + F \sin \theta + \Phi \tag{3.1}$$

Substitution of Eqn. (3.1) into Eqn. (2.1) using Eqn. (2.6) yields the equation for perturbation stream function. It will be:

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2 \Phi + \beta \,\frac{\partial \Phi}{\partial x} + \frac{h_0 \cos \theta}{N^2 (U - C)} \times$$

$$\times \left(k \frac{\partial}{\partial y} - l \frac{\partial}{\partial x}\right) (U \nabla^2 + \beta)\, \Phi = 0 \tag{3.2}$$

where we have neglected higher order perturbation terms.

We would look for a solution of the type :

$$\Phi = \exp\,(i\,\lambda\,t) \sum_{-\infty}^{\infty} P_m \exp\,(i\,\theta_m) \tag{3.3}$$

where,

$$\theta_m = k_m x + l_m y$$

$$(k_m, l_m) = (k_0, l_0) + m\,(k, l)$$

and m takes values in the set of the integers. The solution is similar to solution assumed by Deininger (1981) except that he has chosen $P_{-m} = 0$ for $m \in N$. Our solution will be a physical solution if complex conjugate of Eqn. (3.3) is also a solution of the Eqn. (3.2). Further, a condition, the chosen solution resembles a progressive wave moving in direction (k_0, l_0) and having amplitude that is an even function in space about origin, will be imposed on the solution. Under the condition, we will have :

$$P_{-m} = P_m \tag{3.3a}$$

The recursion relation can be obtained by substitution of Eqn. (3.3) into Eqn (3.1) :

$$h_0 d\, J_{m+1}\, P_{m+1} + (\lambda R_m + k_m J_m)\, P_m +$$

$$+ h_0 d\, J_{m-1}\, P_{m-1} = 0 \tag{3.4}$$

where,

$$d = (k l_0 - l k_0) \,/\, 2N^2 (U - C) \tag{3.5a}$$

$$J_m = U\,R_m - \beta \tag{3.5b}$$

$$R_m = k_m^2 + l_m^2 \tag{3.5c}$$

Substitution of condition (3.3a) for all values of m, in (3.4), we obtain :

$$h_0 d\, J_{-m+1}\, P_{m-1} + (\lambda R_{-m} + k_{-m} J_{-m})\, P_m +$$

$$+ h_0 d\, J_{-m-1}\, P_{m+1} = 0 \tag{3.6}$$

with

$$h_0 d\,(J_1 + J_{-1})\,P_1 + (\lambda R_0 + k_0\,J_0)\,P_0 = 0 \qquad (3.7)$$

where (3.6) is valid for $m \geqslant 1$. Simultaneous solution of (3.4) and (3.6) provides required recursion relation:

$$(\lambda A_m + B_m)\,P_m + h_0 d\,C_m\,P_{m-1} = 0 \qquad (3.8)$$

where $m \geqslant 1$ and

$$A_m = R_m\,J_{-m-1} - R_{-m}\,J_{m+1} \qquad (3.9a)$$

$$B_m = k_m\,J_m\,J_{-m-1} - k_{-m}\,J_{-m}\,J_{m+1} \qquad (3.9b)$$

$$C_m = J_{m-1}\,J_{-m-1} - J_{-m+1}\,J_{m+1} \qquad (3.9c)$$

For $m = 1$, the recursion relation (3.8) becomes :

$$(\lambda A_1 + B_1)\,P_1 + h_0 d\,C_1\,P_0 = 0 \qquad (3.10)$$

Simultaneous solution of (3.7) and (3.10) is used to obtain value of λ through equation :

$$A_1\,R_0\,\lambda^2 + (R_0\,B_1 + A_1\,k_0\,J_0)\,\lambda + [B_1\,k_0\,J_0 - -h_0^2 d^2\,(J_1 + J_{-1})\,C_1] = 0 \qquad (3.11)$$

Then for instability to occur :

$$(R_0\,B_1 - A_1\,k_0\,J_0)^2 + 4\,h_0^2 d^2\,(J_1 + J_{-1})\,A_1 C_1 R_0 < 0 \qquad (3.12)$$

A necessary condition for instability to occur is :

$$(J_1 + J_{-1})\,A_1\,C_1\,R_0 < 0 \qquad (3.13)$$

It can be shown that inequality (3.13) correlates possible values of C/U and N_0^2/N^2 for instability to occur. Here, N_0^2 is defined :

$$N_0^2 = k_0^2 + l_0^2 \qquad (3.14)$$

Further, the mountain height must also exceed a critical value h_c for instability to occur where,

$$h_c^2 = \frac{-(R_0\,B_1 - A_1\,k_0 J_0)^2}{4d^2\,A_1\,C_1\,R_0\,(J_1 + J_{-1})} \qquad (3.15)$$

Now let topography is slightly higher than the critical value by a small amount $\triangle_0$ ($\triangle_0 << h_c$), we have

$$h_c = h - \triangle_0 \text{ and } \triangle_0 = \triangle h_c \qquad (3.16)$$

We also observe that growth rate of perturbation is proportional to $|\triangle_0|^{\frac{1}{2}}$. The result is similar to that obtained by Deininger (1981). The Eqn. (3.2) with use of (3.16) yield :

$$\left(\frac{\partial}{\partial\tau} + U\frac{\partial}{\partial x}\right)\nabla^2\Phi + \beta\frac{\partial\Phi}{\partial x} + + \frac{(h_c + \triangle h_c)\cos\theta}{N^2(U-C)}\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)\times \times (U\nabla^2 + \beta)\,\Phi = 0 \qquad (3.17)$$

4. Non-linear analysis

The Eqn. (3.17) is a complex partial differential equation. We would deal with this equation by a mathematical trick. Let us denote growth of perturbation by :

$$\lambda_c^2 = -2d^2\,(J_1 + J_{-1})\,C_1 \triangle\,h_c^2/A_1\,R_0 \qquad (4.1)$$

We write (3.2) with (3.16) as :

$$(O_1 + \triangle O_2)\,\Phi = 0 \qquad (4.2)$$

where operator O_1 is written as :

$$O_1 \equiv \left(\frac{\partial}{\partial\tau} + U\frac{\partial}{\partial x}\right)\nabla^2 + \beta\frac{\partial}{\partial x} + \frac{h_c\cos\theta}{N^2(U-C)}\times \times\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta) \qquad (4.3)$$

and operator O_2 is written as:

$$O_2 \equiv \frac{h_c\cos\theta}{N^2(U-C)}\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta) \qquad (4.4)$$

Before an attempt is made to develop solution further, it would be useful to discuss the result obtained so far. It is shown that (3.3) is a solution for disturbed stream function to obtain the stream function (3.1) of non-dimensional vorticity equation (2.1). With given parameters and conditions of the problem, the disturbed stream function exhibits a growth provided mountain height exceeds a critical value h_c. Because, solution (3.3) is valid for any height of mountain, the set of solutions can be divided into two subsets. One would corresponds to stable solutions for $h \leqslant h_c$ and other would corresponds to solutions for $h > h_c$ that exhibit growth. It would be interesting to study behaviour of solution when an appropriate expansion for Φ can balance weak effects of non-linearity and instability. Such a study is feasible under assumption (3.16) by expanding disturbed stream function :

$$\Phi = \Phi_0 + \triangle\Phi_1 + \triangle^2\Phi_2 + \ldots\ldots$$

and equating equal power of $\triangle$ in (3.17). It is natural to ask whether such a procedure would assign restriction on magnitude of $\triangle$. The answer would be negative if solution is obtained in principle and Φ_i is shown to be taking finite value in the interval but would be affirmative in practice. Such a study would be useful only if expansion of Φ can be truncated after a few terms. It is believed that if $|\triangle O_2| << |O_1|$, only a few terms would be required to be included in the expansion of Φ. The decision to assign a upper limit for values of $\triangle$ would depend upon the requirement of the problem under consideration. However, it can be expected that a better choice of $\triangle$ would be that which makes order of term involving $\triangle$, least in comparison to order of other terms of the governing equation when a scale analysis is carried out. Now, a solution of Eqn. (4.2) is obtained if it is possible to solve, for $\Phi_0, \Phi_1, \ldots$, the equations :

$$O_1 \Phi_0 = 0 \tag{4.6}$$

$$O_1 \Phi_j = - O_2 \Phi_{j-1} \ \forall j : j \in N \tag{4.7}$$

Let us look for a solution of the type :

$$\Phi_j = \exp(i \lambda \tau) \Sigma P_m^j \exp(i \theta_m) \tag{4.8a}$$

$$\Phi_0 = \exp(i \lambda_r \tau) \Sigma P_m \exp(i \theta_m) \tag{4.8b}$$

where,

$$P_m = P_m^0 \exp(- \lambda_c \tau)$$

The choice of solution (4.8) is similar to that of (3.3) except that λ and coefficients P_m^j's are complex numbers in general. We observe that operator O_1 is similar to the operator that appeared in (3.2). Hence, we solve Eqn. (4.6) by using results that are obtained in the preceding section for a solution of (3.2).

$$\lambda_r = - (R_0 B_1 + A_1 k_0 J_0) / 2A_1 R_0 \tag{4.9}$$

and

$$(\lambda_r A_m + B_m) P_m + h_c d \, C_m P_{m-1} = 0 \tag{4.10}$$

Further, substitution of (4.8) into (4.7) results in a recursion relation :

$$h_c d \, J_{m+1} P_m^j + (\lambda R_m + k_m J_m) P_m^j + + h_c d \, J_{m-1} P_{m-1}^j = f_m^{j-1} \tag{4.11}$$

where,

$$f_m^{j-1} = -d \, (J_{m+1} P_{m+1}^{j+1} + J_{m-1} P_{m-1}^{j-1}) h_c \tag{4.12}$$

Condition, $P_m = P_{-m}$ may be imposed on solution as imposed for linear analysis. The procedure adopted in the preceding section results in recursion relation :

$$(\lambda A_m + B_m) P_m^j + h_c d \, C_m P_{m-1}^j = = -d C_m h_c P_{m-1}^{j-1} \tag{4.13}$$

where $m \geqslant 1$. Further, we obtain a simultaneous condition :

$$h_c d \, (J_1 + J_{-1}) P_1^j + (\lambda R_0 + k_0 J_0) P_0^j = = - d \, (J_1 + J_{-1}) P_1^{j-1} h_c \tag{4.14}$$

Now let

$$\lambda = \lambda_r + i \lambda_c \tag{4.15a}$$

$$P_m^j = P_{mr}^j + i P_{mc}^j \tag{4.15b}$$

for all values of m and j such that $m \geqslant 0$ and $j \geqslant 0$. All $P_{m,r}^j$, $P_{m,c}^j$ are real numbers. The subscript r refers to real part and c refers to complex part. Because, both operators O_1 and O_2 are Hermitian operators :

$$O_1 \Phi_j^* = -O_2 \Phi_{j-1}^* \tag{4.16}$$

is the condition, so that relations (4.8) can be treated as physical solutions. Here an asterisk denotes complex conjugate. The condition (4.16) requires validity of Eqns. (4.11) to (4.14) on equating real and imaginary parts separately. Under the constraint, it may be shown, from (4.13) and (4.15), that

$$\frac{1}{dC_m} \begin{pmatrix} \lambda_r A_m + B_m & - A_m \lambda_c \\ \lambda_c A_m & \lambda_r A_m + B_m \end{pmatrix} \begin{pmatrix} P_{m,r}^j \\ P_{m,c}^j \end{pmatrix} = = - h_c \begin{pmatrix} P_{m-1,r}^{j-1} + P_{m-1,r}^j \\ P_{m-1,c}^{j-1} + P_{m-1,c}^j \end{pmatrix} \tag{4.17}$$

The value of $P_{1,r}$, $P_{1,c}$, $P_{0,r}$, $P_{0,c}$, can be obtained from a simultaneous solution of (4.13) and (4.14) under (4.8), (4.15) and (4.16). We may write in matrix form :

$$\left[\begin{array}{c|c} M_1 & -M_2 \\ \hline M_2 & M_1 \end{array}\right] \left[\begin{array}{c} P_{1,r}^j \\ P_{0,r}^j \\ \hline P_{1,c}^j \\ P_{0,c}^j \end{array}\right] = = \left[\begin{array}{c|c} M_3 & 0 \\ \hline 0 & M_3 \end{array}\right] \left[\begin{array}{c} P_{1,r}^{j-1} \\ P_{0,r}^{j-1} \\ \hline P_{1,c}^{j-1} \\ P_{0,c}^{j-1} \end{array}\right] \tag{4.18}$$

where,

$$M_1 = \begin{pmatrix} \lambda_r A_1 + B_1 & h_0 d \, C_1 \\ h_c d \, (J_1 + J_{-1}) & \lambda_r R_0 + k_0 J_0 \end{pmatrix} \tag{4.19}$$

$$M_2 = \begin{pmatrix} \lambda_c A_1 & 0 \\ 0 & \lambda_c R_0 \end{pmatrix} ;$$

$$M_3 = \begin{pmatrix} 0 & -h_c d C_1 \\ -d h_c (J_1 + J_{-1}) & 0 \end{pmatrix} \tag{4.20}$$

It can be shown that the determinant of the square matrix that appear in (4.18) is non zero. The relation (4.18) can be solved to obtain unique value of P_1^j and P_0^j. The value of P_1^0, P_0^0 can be obtained from (4.8c).

5. Solution analysis

The time dependent topographically forced wave solusion (2.5) for non-dimensional vorticity equation governing the barotropic motion of a quasigeostrophic, inviscid homogeneous fluid on an infinite beta plane, bounded in the vertical direction by an upper flat horizontal plate and by a lower corrugated plane to act as a sinusoidal topography was known. The solution exhibits a singularity in which case the Doppler shifted wave field would be stationary relative to topography. What would be influence of topography on a perturbation ? A detailed study on past attempts to search an answer to the question, reflects a necessity for truncation or assumption on solution to obtain some definite results. The lesser number of deduced algebraic or differential equations in comparison to number of variables is a basic cause for the necessity. It is assumed in this paper that a perturbation is a progressive wave with space dependent amplitude. Further, the amplitude is assumed to be a good function for Fourier series analysis. Such a function can always be written as sum of an even function and an odd function. The constraint on the amplitude to be either an even or an odd function of θ can be used to avoid necessity of truncation (Deininger 1981) to obtain a disturbed stream function. A linear combination of these solutions would also be a solution because perturbed vorticity Eqn. (3.2) is a linear equation.

A review of Eqn. (3.15) shows that value of critical height for instability to occur depends upon value of k_0, k, l_0, l and U. It may be possible to assign a value to these parameters so that instability occurs for any finite height.

The time scale used to derive (2.1) is the advective time L/U. The result of Charney and Devore (1979) suggests that development occurs on the longer time scale $L/(U\triangle^{\frac{1}{2}}\epsilon)$. The growth rate of the perturbation is proportional to $\triangle^{\frac{1}{2}}$ as evident from (4.1). Hence it is in longer time scale T. The non-linear analysis may be carried out in two different ways. One of the ways is replacement of time operator $\partial/\partial t$ by $\partial/\partial t+\triangle^{\frac{1}{2}}\partial/\partial T$ in (2.1) with assumption on Fourier coefficients to be dependent on T. The other way, that is also adopted in the present work, is an assumption of higher order solutions similar to that solution which may have been obtained but for physical constraints on the solution.

6. A comparison with past studies

The present work is an extension of work carried out by Deininger (1981). The present work differs to that of Deininger (1981), Charney and Flierl (1981) etc in choice of possible solution of problem for the perturbed stream function. The past studies have looked for the solution when m in (3.3) takes value in an interval $(0,\infty)$. The present work has assumed a generalized solution in which m takes value in an interval $(-\infty,\infty)$.

The past works suffered either by an assumption on the model and physics involved or by the method of truncation that are to be applied on the solution in order to obtain a close set of algebraic equations and hence, definite results. The present work is free from such, assumptions and limitations.

The growth rate of perturbation is found to proportional to $|\triangle|^{\frac{1}{2}}$. The result was also obtained by past investigators.

Deininger (1981) result speaks about a minimum critical height, non zero for normal flow, for instability to occur. The present work, (3.12) and (3.13) shows that linear instability may occur even for negligible value of mountain height.

The solution chosen by past investigators, (3.3) with or without a restriction on m to be positive integers, mean physically that topography causes simultaneous existence of infinite waves in addition to initial wave in direction $(k_0,\ l_0)$ with same frequency but different wave numbers. However, chosen solution, (3.3) with condition (3.3a) mean physically that topography would only make amplitude of initial wave space dependent. Hence, the present work is in conformity with our understanding of acoustic and light waves.

7. Concluding remarks

In this paper, the linear and non-linear evolution equation were obtained which govern the interaction between a topographical forced wave and its weakly unstable perturbation. The work is carried out primarily on the lines followed by Deininger (1981). The analysis is valid for an even disturbance stream function. On the same lines, an analysis can be carried out for an odd stream function. With that, solution would be completed for disturbance stream functions which are good functions for the Fourier series analysis. Convergence of P_m's can be shown easily on expansion of (3.10). The results obtained in the paper are general except for the choice of basic equation. The right hand side terms of (2.1) differ to that of Pedlosky (1981), hence effect due to external vorticity source and dissipation of vorticity due to the action of Ekman layer are neglected in the present paper. Extension of this analysis to baroclinic model is under progress.

Acknowledgement

The author is thankful to the referee for his useful suggestions.

References

Charney, J.G. and Devore, J.G., 1979, Multiple flow equilibria in the atmosphere and blocking, *J. atmos. Sci.*, **36**, 1205-1216.

Charney J.G. and Flierl, G.R., 1981, *Oceanic analogues of large scale atmospheric motion*: *Evolution of Physical Oceanography*, MIT Press, 504-548.

Charney J.G. and Strauss, D.M., 1980, Form drage instability, multiple equilibria and propagating planetary waves in baroclinic, orographically forced, planetary wave systems, *J. atmos. Sci.*, **37**, 1157-1176.

Deininger, R.C., 1981, Topographically Forced Wave Instability at Finite Amplitude, *J. atmos. Sci.*, **38**, 2619-2625.

Hart, J.E., 1979, Barotropic flow over anisotropic mountains, *J. atmos. Sci.*, **36**, 1736-1746.

Holpainen, E.O., 1970, An observational study of the energy balance of the stationary disturbances in the atmosphere, *Quart. J. R. met. Soc.*, **96**, 626-644.

Holpainen, E.O., 1978, On the dynamic forcing of the long-term mean flow by the large-scale Reynolds' stresses in the atmosphere, *J. atmos. Sci.*, **35**, 1596-1604.

Pedlosky, J., 1970, Finite amplitude baroclinic waves, *J. atmos. Sci.*, **27**, 15-30.

Pedlosky, J., 1979, *Geophysical Fluid Dynamics*, Springer-Verlag, 624 pp.

Pedlosky, J., 1981, Resonant Topographic Waves in Barotropic and Baroclinic Flow, *J. atmos. Sci.*, **38**, 2626-2641.