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On a mathematical model of hailgrowth

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सार — इस शोधपत्र में ओलों की वृद्धि से संबंधित आयतन (वाल्यूम) के परिवर्तन की यांत्रिकी के अध्ययन के लिए वस्तुतः एक गणितीय मॉडल को प्रस्तुत किया गया है। भ्रूण के विभिन्न आकारों के लिए वृद्धि दर के साथ–साथ समयानुसार सांद्रता की विविधता का अन्वेषण किया गया है। जहां यह अध्ययन सू (1983), जीगलर (1983), नेलसन (1983) और हेम्सफील्ड (1982) की प्रेक्षणात्मक खोजों के अनुसार है वहां हिम सांद्रता की विविधता और विशेष रूप से, आरंभिक अवस्थाओं में अर्द्धव्यास में वृद्धि सहित सांद्रता में बूंद के संबंध में कुछ प्रमुख विशिष्ट लक्षणों को प्रकट किया गया है।

ABSTRACT. This paper presents essentially a mathematical model to study the mechanism of the change of volume pertaining to the growth of a hailstone. The variations of concentration with respect to time as well as the growth rate for different sizes of embryo are investigated. While this study is in accord with the findings of Xu (1983), Ziegler (1983), Nelson (1983) and Heymsfield (1982), it brings out some important characteristic features in respect of the variations of ice concentration and particularly, a drop in the concentration with increase in radius in initial stages.

1. Introduction

The growth of hailstone is considered as one of the important ancillary developments in the formation and growth of convective storms, while there appear to be few studies and that, too, of an observational type. As far as the present authors are aware, there does not exist any significant study on hailstone in the Indian context. Although studies on hailstone date back to the work of Schuman in thirties with a statistical content, it is only in the 80's that there was a new spurt for such investigations. Most of these studies, as already mentioned, deal with observations. For example, Heymsfield's (1982) concern was with that phase of hailgrowth when the entire quantity of water collected without being frozen immediately is either soaked into the interior of the particle or is accumulated on the surface due to various cloud conditions. Nelson's (1983) was an experimental model study and pertinently it tells us about the initial growth rate of the hailstone, particularly in the wet phase of growth. Xu (1983) developed a three dimensional cloud model to study the growth of a hailstone with in a supercell storm, the growth being taken as a summation of melting rate, the depositional growth rate and the accretion of ice particles. The growth in this model was found to depend on embryo radius, density and cloud droplet concentration. Xu (1983) has also found that smaller embroys require more time to grow into a hailstone. Of all observational studies Ziegler *et al*. (1983) have made a significant dent. Ziegler *et al*. (1983) studied hailstone growth within an Oklahoma multicellular storm. The observed hail trajectories were then compared with a numerical Lagrangian growth model. Their findings show that a longer growth time is necessary for smaller embryos, which conform the findings of Xu (1983). Embryo injection into the main updraft provides the bulk of large hail production. According to Ziegler *et al*. (1983), the overall hailgrowth is dominated by wind field characteristics.

In all the foregoing studies, the aspect of variation of ice concentration which plays an important role in the growth of hailstone has not been taken into account. The purpose of the present paper is to consider this fundamental aspect within a mathematical frame-work. Thus, a time dependent hailgrowth model has been developed in mathematical terms. The model is used to study the growth of a hailstone and the variations of ice concentration in it. The dependence of other microphysical parameters on concentration and their effects on growth have also been studied. The present model does not provide for the dynamical/microphysical interaction.

The analysis presented in this paper enables us to arrive at some conclusive results which do agree with observational findings of Heymsfield (1982), Nelson (1983). Xu (1983) and Ziegler *et al*. (1983). An important spin-off of this modelling exercise in mathematical terms is that there exists some significant variation in respect of ice concentration dependent on time and radial distances. The drop or the discontinuity in the growth processes, particularly, at the boundary between the two phases is also well brought out in this presentation. The features

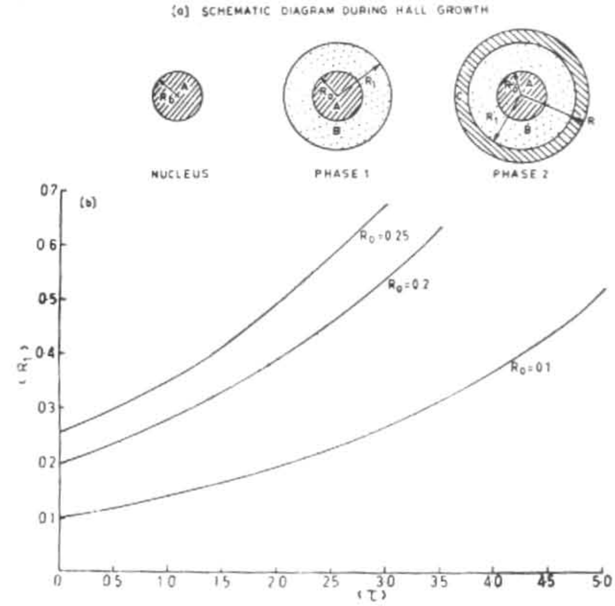
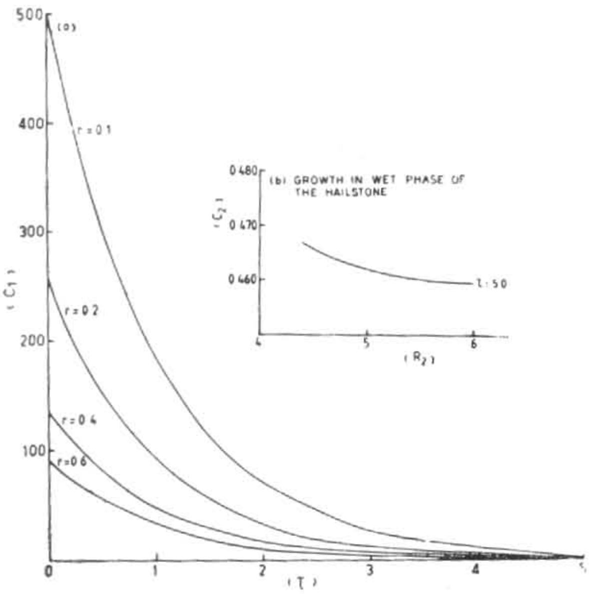


Fig. 1. (a). Schematic diagram during hailgrowth and (b). Variation of growth of hailstone in dry phase with time for different embryo radii



Figs. 2 (a&b). Variation of ice concentration in (a) dry phase with time at different radial distances and (b) with growth in wet phase of the hailstone

of such growth processes especially those relating to the sizes and different from what have already been considered by previous workers show that the mathematical model provides a better understanding of the phenomena of growth processes as a whole of the hailstone. The exercise undertaken in this paper is reinforced by graphs based on realistic data. It is believed that such conclusions will be of use in the understanding of convective storm studies.

2. Model design

The schematic diagram during the hailgrowth in the present model is shown in Fig. 1(a).

The growth of the hailstone is assumed to take place in two phases. The first phase begins from an initial core stage, called the embryo with radius R_0 having concentration C_0. On account of the extremely small dimension of the embryo, C_0 has been assumed to be constant. The temperature of the hailstone is below 0°C in this phase, which indicates that the entire quantity of water accreted in this phase freezes and the growth is a dry growth. It can, therefore, be said that the embryo is surrounded by a layer of dry growth whose ice concentration $C_d > \bar{C}$, where $\bar{C}$ is the critical ice concentration below which melting starts. This phase of dry growth ends when $C_d = \bar{C}$. During the next phase of hailgrowth, the entire quantity of water collected does not freeze immediately and either it is soaked into the interior of the particle or is accumulated on the surface due to various cloud conditions (Heymsfield 1982). It is thus indicated that in the second phase of hailgrowth, the embryo and the layer of dry growth are surrounded by a layer of wet growth of ice concentration C_w, where $C_w < \bar{C}$.

3. Mathematical formulation of the model

The concentration of ice in the dry phase of hailstone is given by :

$$\frac{\partial C_d}{\partial t} = \frac{K}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial C_d}{\partial r}\right) ; \; R_0 \leqslant r \leqslant R_1 \qquad (1)$$

where,

K=bulk density of the particle,

C_d=concentration of ice in the dry phase of hailgrowth,

r=radial coordinate,

R_0=radius of embryo,

R_1 = radius of dry phase,

t = time

The growth of the dry phase is given by equation :

$$\frac{4}{3}\pi \frac{d}{dt}(R_1^3) = \frac{4}{3}\pi S R_1^3 \qquad (2)$$

where, S=normal rate of growth of hailstone in dry phase per unit volume per unit time.

The initial condition is defined as :

$$t=0, \; r=R_0, \; C_d=C_0 \qquad (3)$$

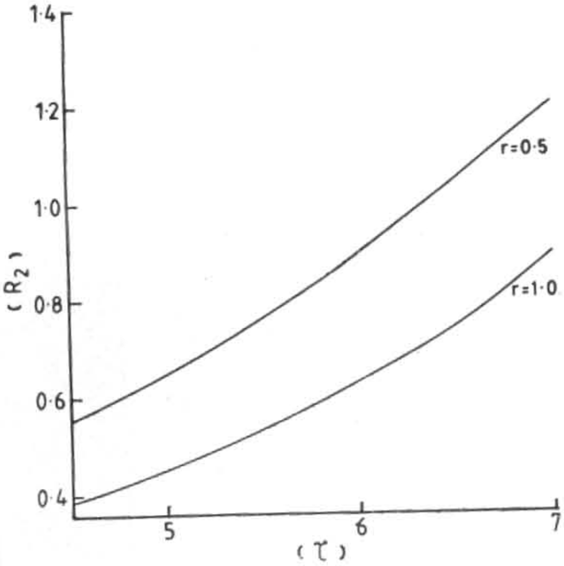


Fig. 3. Variation of growth of hailstone in wet phase with time at different radial distances

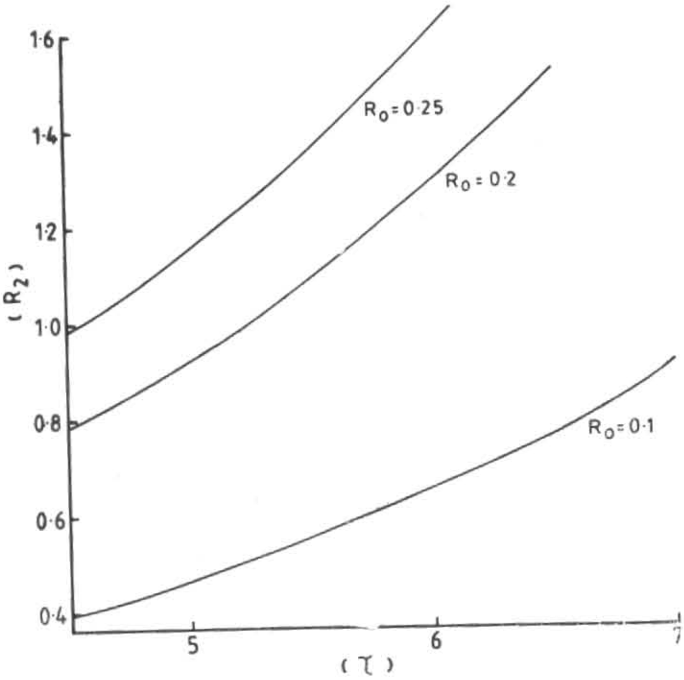


Fig. 4. Variation of growth of hailstone in wet phase with time for different embryo radii

The ice concentration in dry phase, C_d and the growth of hailstone in this phase, R_1 are obtained by solving Eqns. (1) and (2) alongwith condition (3), as :

$$\bar{C}_d = \frac{\bar{C}_0 \bar{R}_0}{\bar{r}} e^{-\bar{\alpha}\tau}\left[\cos\sqrt{\bar{C}}(\bar{r}-\bar{R}_0) + \frac{1}{\bar{R}_0\sqrt{\bar{C}}}\sin\sqrt{\bar{C}}(\bar{r}-\bar{R}_0)\right] \tag{4}$$

where, $\bar{C}_d$, $\bar{C}_0$, $\bar{R}_0$, $\bar{r}$, $\bar{\tau}$, $\bar{\alpha}$ are dimensionless forms of C_d, C_0, R_0, r, τ and α respectively,

and $\bar{R}_1 = \bar{R}_0 \exp(\tau/3)$ (5)

where, $\bar{R}_1$ is dimensionless form of R_1.

The ice concentration, C_w, of the wet phase of hailgrowth is given by :

$$\frac{\partial C_w}{\partial t} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C_w}{\partial r}\right) - PC_w; \quad R_1 \leqslant r \leqslant R_2 \tag{6}$$

where, C_w=concentration of ice during the wet phase of hailgrowth,

R_2 = radius of wet phase,

and P = rate of collection of ice particles.

The growth of the wet phase of hailstone is given by:

$$\frac{d}{dt}\left(\frac{4}{3}\pi R_2^3\right) = \frac{4}{3}\pi S R_1^3 + (S-\lambda)\frac{4}{3}\pi R_2^3 \tag{7}$$

where, λ = local rate of volume loss per unit volume per unit time.

The conditions on the boundary of different phases of the hailstone are :

$$r = R_1, \; C_d = C_w$$
$$r = R_2, \; \frac{dC_w}{dr} = 0 \tag{8}$$

The concentration C_w of ice during the wet growth and the growth of hailstone during this phase is given by :

$$\bar{C}_w = \frac{\bar{C}_0 \bar{R}_0 \bar{R}_1}{\bar{r}^2}\left[\exp(-\bar{\alpha}\tau)\left\{\cos\sqrt{\bar{C}}(\bar{r}-\bar{R}_0) + \frac{1}{\bar{R}_0\sqrt{\bar{C}}}\sin\sqrt{\bar{C}}(\bar{r}-\bar{R}_0)\right\}\right] \times$$
$$\times \frac{\left[\sqrt{\alpha-P}\cos\sqrt{(\alpha-P)\bar{C}/\alpha}\,(\bar{r}-\bar{R}_2) + \frac{1}{\bar{R}_2}\sqrt{\alpha/\bar{C}}\sin\sqrt{(\alpha-P)\bar{C}/\alpha}\,(\bar{r}-\bar{R}_2)\right]}{\left[\sqrt{\alpha-P}\cos\sqrt{(\alpha-P)\bar{C}/\alpha}\,(\bar{R}_1-\bar{R}_2) + \frac{1}{\bar{R}_2}\sqrt{\alpha/\bar{C}}\sin\sqrt{(\alpha-P)\bar{C}/\alpha}\,(\bar{R}_1-\bar{R}_2)\right]} \tag{9}$$

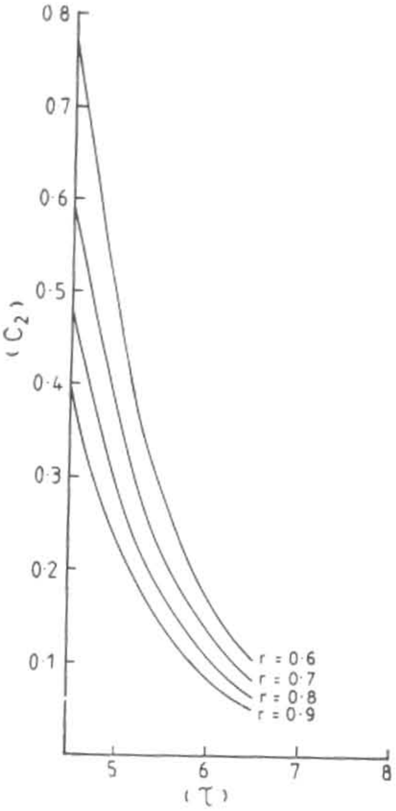


Fig. 5. Variation of ice concentration in wet phase of hailstone with time at different radial distances

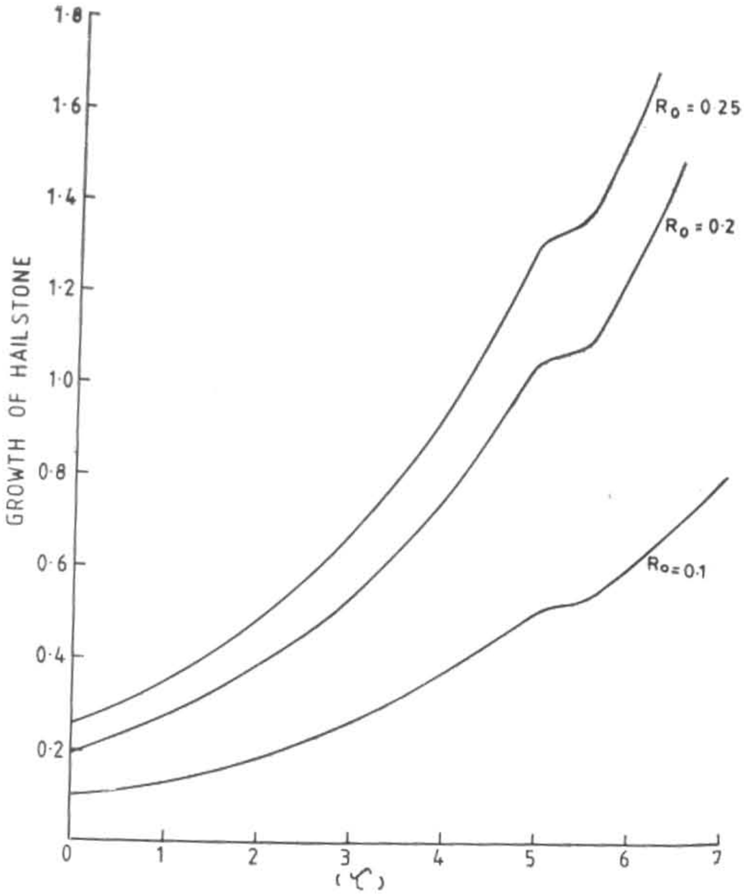


Fig. 6. Variation of total hailgrowth with time for different embryo radii

TABLE 1

Variation of ice concentration in dry phase with time at different radial distances for constant values of $\alpha=1$ and $R_0=0.1$, $C_0=500$

τ	$\bar{r}$	$\bar{C}_d$	τ	$\bar{r}$	$\bar{C}_d$
0	0.1	500	2	0.3	25.20
—	0.2	245.36	—	0.4	17.80
—	0.3 (0.3)	172.47	—	0.5	14.75
—	0.4	131.54	—	0.6	12.26
—	0.5	106.98	3	0.1	24.89
—	0.6	90.60	—	0.2	12.66
1	0.1	183.94	—	0.4	6.55
—	0.2	93.29	—	0.6	4.51
—	0.3	48.39	—	0.8	3.49
—	0.4	33.33	4	0.1	9.16
—	0.5	25.80	—	0.2	4.66
—	0.6	21.28	—	0.4	2.41
2	0.1	67.67	—	0.6	1.66
—	0.2	34.42	—	0.8	1.28

TABLE 2

Variation of growth of hailstone in dry phase with time for different embryo radii

$\bar{R}_0$	τ	$\bar{R}_1$	$\bar{R}_0$	τ	$\bar{R}_1$
0.1	0	0.1	0.2	3	0.540
	1	0.14		4	0.760
	2	0.195		5	1.060
	3	0.272	0.25	0	0.25
	4	0.379		1	0.35
	5	0.530		2	0.49
0.2	0	0.20		3	0.68
	1	0.279		4	0.95
	2	0.390		5	1.32

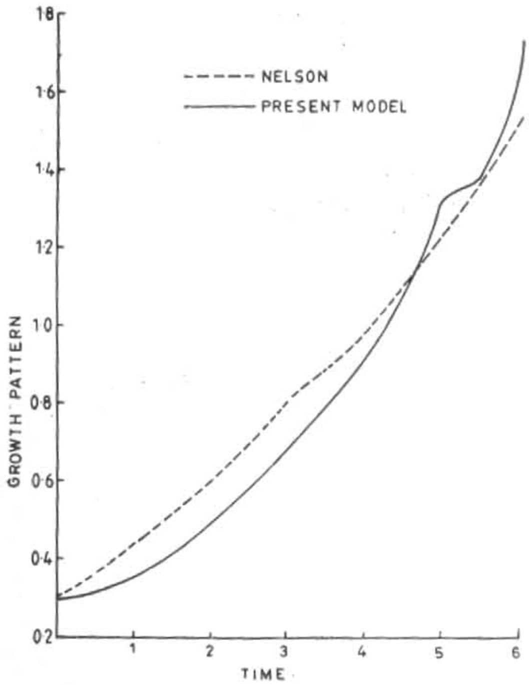


Fig. 7. Comparison of growth pattern of Nelson with present model

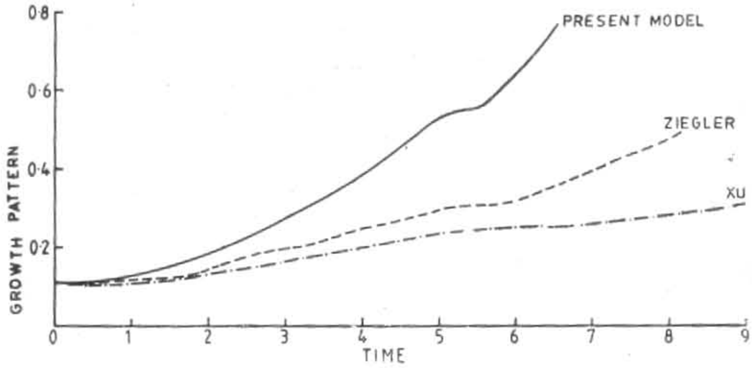


Fig. 8. Comparative study of different hailgrowth pattern

TABLE 3

Variation of ice concentration in wet phase of hailstone with time at different radial distances for constant values of C_0=500, R_0=0.1, R_1=0.45

τ		$\bar{r}$	$\overline{C_W}$	τ	$\bar{r}$	$\overline{R_2}$	$\overline{C_W}$
4.5	0.6	0.45	0.775	6.0	0.6	0.65	0.682
	0.7	0.45	0.588		0.7	0.65	0.132
	0.8	0.45	0.465		0.8	0.65	0.104
	0.9	0.45	0.380		0.9	0.65	0.085
5.0	0.6	0.46	0.475	6.5	0.6	0.76	0.105
	0.7	0.46	0.363		0.7	0.76	0.080
	0.8	0.46	0.282		0.8	0.76	0.063
	0.9	0.46	0.230		0.9	0.76	0.050

TABLE 4

Variation of growth of hailstone in wet phase with time for different embryo radial and constant value of γ=1.5

$\overline{R_0}$	τ	$\overline{R_2}$	$\overline{R_0}$	τ	$\overline{R_2}$
0.1	4.5	0.39	0.2	5.5	1.09
	5.0	0.46		6.0	1.29
	5.5	0.55			
	6.0	0.65	0.25	4.5	0.98
				5.0	1.16
0.2	4.5	0.78		5.5	1.37
	5.0	0.93		6.0	1.60

and

$$\overline{R}_2^3 = \frac{1}{\gamma}\,\overline{R}_0^3 e^{\tau} + \overline{R}_0^3\left(1-\frac{1}{\gamma}\right)e^{(1-\gamma)\tau} \qquad (10)$$

where, $\gamma=\lambda/S$, is a constant, $\bar{C}_w$ and $\bar{R}_2$ are respectively the dimensionless form of C_w and R_2.

4. Results and discussions

The ice concentration of hailstone in dry phase, $\bar{C}_d$ and in wet phase, $\bar{C}_w$ as well as the growth pattern of the hailstone in both the phases are given, in Eqns. (4), (9) and (5), (10) respectively. These equations are used to study the volume change mechanism during growth of the hailstone. Embryos of three different sizes, *viz.*, of radii 0.1 cm, 0.2 cm and 0.25 cm are considered in the present study whereas embryo concentration is taken to be of constant value, C_0= 5E—4 M^{-3}. The value of the embryo density is kept to be 0.91 gm/cm³ as Xu (1983). The critical concenration $\bar{C}$ is assumed to be 1. $\gamma=\lambda/S$ is arbitrarily fixed at 0.5, 1.5 and 2.0. Since γ is the ratio of the local volume loss to volume gain of ice particles, therefore, for $\gamma<1$, the growth of the hailstone refers to the dry growth and for $\gamma>1$, the growth would be a wet growth. Fig. 1(b) shows the change of volume of the hailstone for different values of τ in the dry phase whereas the change of volume in the wet phase is shown in Fig. 3. It is observed in Figs. 1 (a) and 4 that a longer time is necessary for smaller embryos to grow into hailstones. This result is in accord with that of Ziegler (1983) and Xu (1983). It can be concluded from Fig. 6 that smaller the sizes of embryos, the longer is the time to grow. This analysis further shows embryos of bigger radii grow into larger hailstones, which again conform to findings of Ziegler *et al.* (1983).

Let us now take up some new features which are not in agreement with those of Nelson (1983), Xu (1983), Ziegler *et al.* (1983) and Heymsfield (1982).

The growth pattern of the hailstone obtained from the present mathematical model is compared with the models of Xu, Ziegler and Nelson in Figs. 7 and 8. It is observed that the hailstone takes a longer time to grow in the models of Xu and Ziegler than the present model. The discontinuity shown at the boundary between the dry and wet phases during the hailgrowth in the present model is weakly exhibited in the models of Xu and Ziegler, whereas the growth rate of the hailstone in the present model and that in the model of Nelson are nearly similar except in the fact that the discontinuity is not clearly revealed in Nelson's model. The initial growth rate obtained from the present model is found to be less than that of the model of Nelson. Further it increases and then exceeds that of Nelson's model during the wet phase of growth. At about radius equal to 1.1 cm, there appears to be a common point between the present growth pattern and Nelson's growth.

The ice concentration in the two phases, shown in Figs. 2 (a) and 5, decrease sharply with time at different radial distances from the core. The drop (fall) in the concentration with increasing radius is more pronounced at the initial time step, than what it is at subsequent time and it approaches 1, the limit of the dry phase. The break during the growth of the hailstone at the boundary of dry and wet phase may be attributed to the fact that at the beginning of the wet growth unfrozen liquid water accumulated by the hailstone might be absorbed into the hailstone to fill in the crystal branch spaces. Thus, during this period there is, perhaps, practically no increase in the volume during the growth of the hailstone (Heymsfield 1982). We can further add that the concentration of large hailstones is less than that of smaller hailstones at any given time (Fig. 2b).

Acknowledgement

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