

## An application of finite element technique for solving P.B.L. equations\*

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**सार** — मौसम विज्ञान प्राचलों के अवकलनात्मक समीकरणों को हल करने के उद्देश्य से परिमित अन्तर विधि का उपयोग किया गया और ये संगणकों (कम्प्यूटरों) को उपलब्ध कराए गए। इस शोध पत्र में, वायुमंडलीय गतिमान परिसीमित स्तर (पी बी एल) समीकरणों के हल के लिए परिमित तल तकनीक का प्रयोग किया गया। इस तकनीक द्वारा प्राप्त परिणाम, परिमित अन्तर विधि के परिणामों के साथ परिशुद्धता में तुलनीय है।

**ABSTRACT.** In order to solve the differential equations of meteorological parameters finite difference method is used and fed to computers. In this paper finite element technique is used for the solution of atmospheric planetary boundary layer (PBL) equations. The results obtained by this technique are comparable in accuracy with those obtained by finite difference method.

### 1. Introduction

With the advent of computer, there has been tremendous advances in obtaining solutions for partial differential equations occurring in the field of atmospheric science. So far, the main thrust of the development of the numerical models in the meteorological field has been with the finite difference method. However, other engineering fields especially aircraft technology has used the finite element method to obtain solutions. The purpose of this paper is to develop a finite element application for the solution of atmospheric planetary boundary layer equations. The preliminary objective is to learn the characteristics of a finite element model and compare it with a similar finite difference model. The physical phenomenon proposed to be studied corresponds to that of a heated island effects.

We present in this preliminary study, the solutions for a homogeneous terrain, *i.e.*, homogeneous with respect to temperature roughness etc calculations are in progress to include nonlinear effects, variation of roughness etc. The relevant equations are solved using Galarkin residual procedure of finite elements instead of the usual finite difference technique hitherto employed.

### 2. Finite elements and resumé of the numerical procedure

The basic concept of the finite element method is that a solution can be accurately determined by a

sum of simple easily computable functions. The Galarkin procedure involves subdividing the domain into a set of elements, approximating the dependent variables by a linear combination of low order polynomials (having value zero except over a particular element), inserting these approximations into the equation, multiplying the equation by a test function (also a low order polynomial having a one to one correspondence with the basic function) whose purpose is to minimize, the residual between the approximate solution and the actual solution, integrating over the entire domain and finally solving the system of equations assembled during the integration for the solution. Better or finer resolution for the dependent variables at any particular region is obtained by varying the size/shape of the finite element at will. In the finite difference technique finer resolution in the region where the variables have a large gradient is obtained by varying grid size by logarithmic transformation of a coordinate and/or similar transformations. This is usually a cumbersome process.

In this study, both finite difference and finite element techniques were employed to check and verify the correctness of the calculations.

### 3. Basic equations

The vertical distributions of wind, temperature and moisture over a homogeneous terrain can be obtained

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from the solution of the following differential equations:

$$\begin{aligned} -fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right) \\ fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right) \\ \frac{\partial}{\partial z} \left( K \frac{\partial \theta}{\partial z} \right) &= 0 \\ \frac{\partial}{\partial z} \left( K \frac{\partial Q}{\partial z} \right) &= 0 \end{aligned}$$

where the symbols carry the usual meaning. The pressure gradient terms,  $\partial p/\partial x$ ,  $\partial p/\partial y$  which specify the nature of large scale flow can vary with altitude. By specifying appropriately the eddy coefficient  $K$  (assumed to be the same for heat, momentum and moisture transfer) we can obtain analytical solutions. With appropriate boundary conditions and for constant  $K$  the above equations yield Ekman solutions. However, in the real atmosphere, the eddy coefficient is a function of altitude, stability parameter, roughness length and shear of the basic wind. The specification of  $K$  including the above mentioned parameters is not unique. For our calculation, we have used Blackadar (1962) formulation as below:

$$\begin{aligned} K &= l^2 \frac{\partial V}{\partial z} \left( 1 + \alpha R_i \right), \quad R_i > 0 \\ &= l^2 \frac{\partial V}{\partial z} \left( 1 - \alpha R_i \right)^{-1}, \quad R_i \leq 0 \end{aligned}$$

where,  $l = k_0(z + z_0) / [1 + k_0(z + z_0)/\lambda]$   
 $\lambda = 0.00027 v_g / f$ ,  $V = u^2 + v^2$ ,  $\alpha = -3.0$   
 $k_0 = 0.4$  and  $z_0 = 0.1$

Since the eddy coefficient is assumed to be positive the absolute value of  $\partial V/\partial z$  is used. However, we have found in our experience, in order to have a smooth variation of  $K$  over the entire domain  $\partial V/\partial z$  should be replaced by the expression  $\left\{ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right\}^{1/2}$ .

The scientific reason for using the above expression is also given by Torrance *et al.* (1971). For the simple case with  $K$  constant, Galerkin's technique of weighted residuals yield (for the first two equations) the following equations:

$$\begin{aligned} \int N_i \left\{ K \frac{\partial^2 u}{\partial z^2} + fv \right\} dz &= 0 \\ \int N_i \left\{ K \frac{\partial^2 v}{\partial z^2} - fu + fu_g \right\} dz &= 0 \end{aligned}$$

where  $N_i$  is the test function and integration is over a finite element. The dependent variables  $u$ ,  $v$  are now represented over a particular finite element by the following expressions:

$$\begin{aligned} u &= \sum N_j u_j \\ v &= \sum N_j v_j \end{aligned}$$

where,  $N_j$  represents a piecewise continuous function also known as basis function. In this investigation we have taken both the test and basis function identical and a quadratic polynomial in  $z$  (independent variable) is used to represent them. Inserting the above expressions into the integrals, integrating by parts we obtain for  $i^{\text{th}}$  node

$$\begin{aligned} F_1^{ij} u_j + F_2^{ij} v_j &= 0 \\ F_3^{ij} u_j + F_4^{ij} v_j &= F_4^{ij} \end{aligned}$$

where,

$$\begin{aligned} F_1^{ij} &= -\int K \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} dz \\ F_2^{ij} &= \int N_i f N_j dz \\ F_3^{ij} &= -F_2^{ij} \\ F_4^{ij} &= -\int N_i f v_g dz \end{aligned}$$

In our calculations, we have taken 39 line elements each with three nodal points. To have a better resolution lengths of elements adjacent to earth's surface are taken to be small and at further distances these element lengths are increased as shown in Fig. 1. Since each basis and test function is zero over the domain except over a particular element, the global integration can be performed by integrating locally over each element. Repeating the procedure for  $N$  grid points ( $N$  elements) a system of  $N$  equations with  $N$  unknowns are generated. Thus for the entire domain under consideration we can write the basic equations in a compact form:

$$[F][V] = [D]$$

where  $[F]$  is a matrix consisting of  $F_1, F_2, F_3, F_4$  expressions defined above,  $[D]$  and  $[V]$  are column vectors consisting of constants and unknown variables  $u$  and  $v$ . The imposed boundary conditions are included in the column vector  $[D]$ .

We have imposed the following boundary conditions for the problem:

$$\begin{aligned} u = 0, v = 0, \theta = \theta_0, Q = Q_0 \quad \text{at } z = 0 \\ u = u_g, v_g = 0, \theta = \theta_H, Q = Q_H \quad \text{at } z = H \end{aligned}$$

where,  $H$  is the height of planetary boundary layer,  $fu_g = -\frac{1}{\rho} \frac{\partial p}{\partial y}$  and  $f$  is coriolis parameter. In a similar manner, now including the variation of  $K$  with height the basic equations can be written:

$$\begin{aligned} \int N_i \left\{ \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right) - fv \right\} dz &= 0 \\ \int N_i \left\{ \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right) - fu + fu_g \right\} dz &= 0 \end{aligned}$$

which can be reduced as before to a matrix form:

$$[F][V] = [D]$$

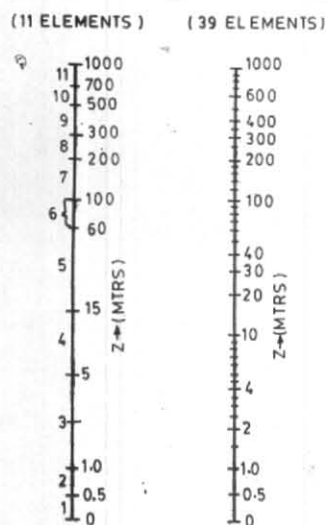


Fig. 1

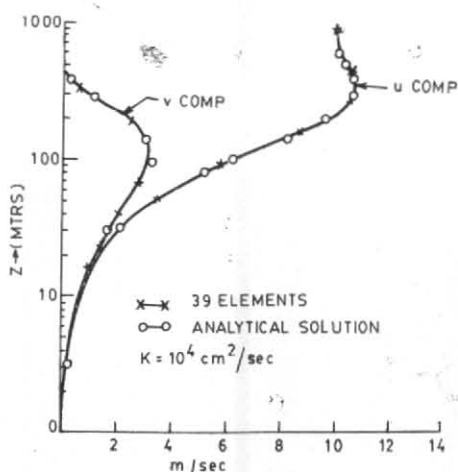


Fig. 2

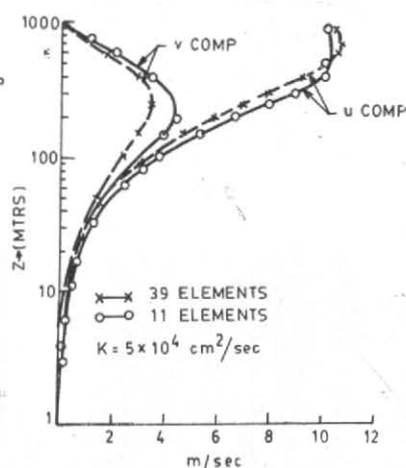


Fig. 3

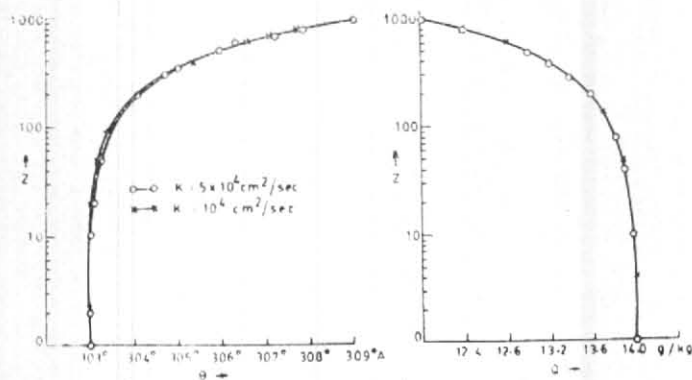


Fig. 4

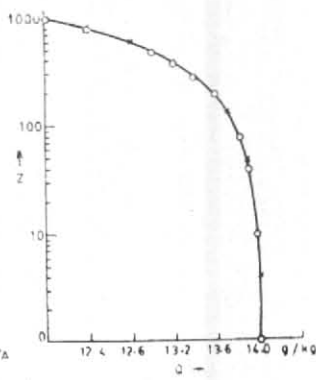


Fig. 5

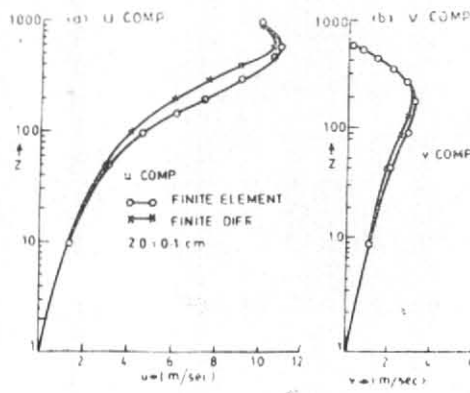


Fig. 6(a)

Fig. 6(b)

Since  $K$  is a variable and function of  $z$  the matrix coefficients containing  $K$  should be evaluated at all nodal points to include shear of basic wind and the stability parameter,  $R_i$ . The same interpolation function  $N_i$  was used for  $K$  in this analysis.

An identical approach is made use of to evaluate temperature and humidity profile in the entire domain. With  $K$  as constant, the temperature and humidity equations can be solved separately. However, with  $K$  as a function of stability parameter and the shear of the basic wind for each iteration  $u, v$  components are evaluated first and substituting these in the  $K$  expression temperature and humidity equations are solved. The Richardson number ( $R_i$ ) is assumed to be constant with height and equal to the average value between surface and 100 metres.

#### 4. Results and discussion

For computations we have used the following values for the variables :

$$\begin{aligned} u_g &= 1000 \text{ cm/sec;} \\ f &= 0.98 \times 10^{-4} \text{ sec}^{-1} \\ v_g &= 0; \\ \theta_0 &= 303^\circ\text{A}; \quad \theta_H = 309^\circ\text{A}; \\ Q_0 &= 14 \text{ g/kg;} \\ Q_H &= 12 \text{ g/kg.} \end{aligned}$$

To test the validity of finite element technique few runs were made with different values of  $K$  which can vary over large range. By varying the size of finite elements the accuracy of adopted technique is tested. Typical

results are given in accompanying figures. Fig. 1 shows the structure of line elements in the domain considered. In our simplest case, the elements are simple line elements also colinear. Fig. 2 and Fig. 3 show the increase in accuracy of calculations with increase in mesh size for  $K = 5 \times 10^4$  cm<sup>2</sup>/sec and  $K = 10^4$  cm<sup>2</sup>/sec. For comparison purposes results obtained using analytical formulae are also shown.

The temperature and humidity profile obtained with different values of  $K$  are shown in Figs. 4 & 5. The results obtained using finite element technique and finite difference method is compared in Fig. 6.

#### 5. Summary and conclusion

With the results obtained it is clear finite element technique has comparable accuracy with the finite difference and analytical results. The calculations show the versatility of the adopted technique for application to atmospheric science problems, calculations are in progress with the formulation to include

advection terms in our equations. However, it has been found in our experience finite element technique require more computational time than finite difference technique.

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