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Use of linearised quasi-geostrophic numerical model for evaluation of truncation errors*

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सार—एक रेखीय अर्द्ध-भूव्यापी बहुपरत, संख्यात्मक निदर्श का माध्य मानसून क्षेत्रीय गति के बेरोक्लीनिक अस्थिर प्रणाली विन्यास के परीक्षण हेतु विकास (मिश्र और सालवेकर 1980) में एक प्रारम्भिक अध्ययन किया गया है। इसी निदर्श का उपयोग बैरोक्लिनिक अस्थाई तरंग तथा प्रावस्था वेग के उर्ध्वाधर तथा समय खंडन त्रुटियों को प्राप्त करने के लिये किया गया है। यह बैरोक्लिनिक अस्थाई तरंग परिमित अन्तर सन्निकटन के कारण पैदा होती है।

ABSTRACT. A linear, quasi-geostrophic, multilayer, numerical model which has been developed in an earlier study (Mishra & Salvekar 1980) for investigating baroclinic instability mechanism of mean monsoon zonal flow, is used here to obtain vertical and time truncation errors in growth rate and phase velocity of a baroclinic unstable wave that arise due to finite difference approximations.

1. Introduction

The general governing equations for atmospheric motions are non-linear in nature, whose analytic solutions are not known except for a few very simplified cases of one-dimensional motion. In general, even for linearised system of equations analytic solutions are not always possible because solutions of differential equations are known only for the cases when their coefficients are constants or have some particular functional forms. It is obvious from above discussion that numerical techniques are indispensible for obtaining the solutions of linear and non-liear equations of various atmospheric problems. For the purpose of obtaining numerical solutions, the differential equations are replaced by corresponding set of difference equations. But such procedure always involves truncation errors.

The accuracy of various predicted meteorological parameters obtained from time integration of Numerical Weather Prediction model is limited by the presence of truncation errors in the model in addition to the other physical factors. For a meaningful prediction of the atmospheric motions by the models require a rough estimate of truncation errors. Estimation of truncation errors as their first approximation can be obtained with the help of linear models. Wiin Nielsen (1962) has obtained vertical truncation error for a quasi-geostrophic linear model. Gates (1959) found horizontal and time truncation error involved in a numerical solution of barotropic vorticity equations. Rosenthal (1964) also compared analytical and numerical solutions by integrating the quasi-geostrophic linear model for simple cases. All these studies are pertaining to the mid latitude atmosphere. But no such study is made so far for the disturbances embedded in the tropical zonal flow.

Linear numerical models can be used for the study of dynamic instability, estimation of numerical trucation errors in space and time and propagation of initial error in time. In the present study the quasi-geostrophic (Q.G.), numerical, linear model, which is developed for baroclinic instability study of mean monsoonal zonal flow by Mishra and Salvekar (1980), hereafter referred as MS, is used to obtain vertical truncation errors in growth rate and phase velocity. General procedure for obtaining vertical and time truncation error is discussed and percentage errors in

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phase velocity due to vertical and time truncation are obtained explicitly.

2. The Q.G. numerical model

The Q.G. numerical linear model used in the present study is described in detail by MS where a wave in zonal direction superimposed on a basic zonal flow $\bar{U}$ was considered. The numerical model is based upon the potential vorticity equation. The atmosphere is divided into N layers in vertical each of uniform thickness in pressure ($\triangle p = p_0/N$). The basic zonal flow $\bar{U}$ and stream function ψ' are defined in the middle of layers while at the levels static stability σ is considered. The governing linearised potential vorticity equation (in standard notations) :

$$\left[\nabla^2 + f_0^2 \frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial}{\partial p}\right)\right]\frac{\partial \psi'}{\partial t} + \left[\bar{U}\nabla^2 + \beta_0 - f_0^2 \frac{d}{dp}\left(\frac{1}{\sigma}\frac{d\bar{U}}{dp}\right)\right]\frac{\partial \psi'}{\partial x} + f_0^2\, \bar{U}\frac{\partial}{\partial p}\left[\frac{1}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial \psi'}{\partial x}\right)\right] = 0 \qquad (1)$$

is applied at the middle of each layer using centred difference scheme to the pressure derivatives, and the boundary condition :

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial p} = 0 \text{ at } p=0 \text{ and } p=p_0=1000 \text{ mb} \qquad (2)$$

is incorporated while applying Eqn. (1) at the top and bottom layer. By this numerical procedure the potential vorticity equation, which is a partial differential equation, reduced to a system of $2N$ coupled ordinary differential equations in $\psi_1(1), \psi_1(2), \ldots, \psi_1(N)$ and $\psi_2(1), \psi_2(2), \ldots, \psi_2(N)$ which can be written in matrix notation as

$$B\frac{d}{dt}\begin{bmatrix}\psi_L(1)\\ \psi_L(2)\\ \vdots\\ \psi_L(N)\end{bmatrix} = (-1)^{LL}.k.A\begin{bmatrix}\psi_{LL}(1)\\ \psi_{LL}(2)\\ \vdots\\ \psi_{LL}(N)\end{bmatrix} \qquad (3)$$

where $L=1, 2$ and $LL=3-L$. The A and B are $N \times N$ matrices depending on basic state parameters, 'k' is the zonal wave number ($k=2\pi/\lambda$, λ=zonal wavelength) and $\psi_L(1), \ldots, \psi_L(N)$ are amplitudes of the wave perturbation of sine and cosine waves at each layer. Since the linearised model is considered, the system of differential equations can easily be solved by inverting the matrix 'B' instead of using the time consuming iterative scheme, which is rather essential for non-linear models. Time integration of the model is performed using 'Modified Euler Backward' scheme for the first time step and 'Leap-frog' scheme for the subsequent time steps.

The model has been used successfully for the study of baroclinic instability of Indian monsoonal zonal flow (MS). It has been also shown that 20 layers in vertical are sufficient for this purpose. In this numerical model, the perturbation vertical velocity is obtained by applying thermodynamic energy equation at the levels. Its finite difference form yields ($N-1$) ordinary differential equations which can be written in matrix form as :

$$\begin{bmatrix}\omega_L(1)\\ \omega_L(2)\\ \vdots\\ \omega_L(N-1)\end{bmatrix} = C\begin{bmatrix}\frac{\partial \psi_{LL}}{\partial t}(1)\\ \frac{\partial \psi_{LL}}{\partial t}(2)\\ \vdots\\ \frac{\partial \psi_{LL}}{\partial t}(N)\end{bmatrix} - D\begin{bmatrix}\psi_L(1)\\ \psi_L(2)\\ \vdots\\ \psi_L(N)\end{bmatrix} \qquad (4)$$

where C and D are $(N-1) \times N$ matrices depending on basic state parameters. Perturbation temperature field at levels is obtained from the hydrostatic balance.

3. Procedure for estimation of truncation errors

The general procedure for estimation of truncation error is to find the difference of the solution of diffedential equation and that of corresponding difference equation. But this direct method is not applicable for non-linear equations as well as for some linear equations also, because analytical solutions are available only for few simple linear equations. Therefore, we have to follow the indirect method. As the grid length (space and time) decreases the numerical solution becomes closer to that of analyticals solution.

3.1. *Time truncation error*

Analytical solution of Eqn. (1) is possible only for the simple case when basic parameters does not change with height. But the actual atmospheric situation are not so simple. In the case of monsoon, situation, vertical profile of $\bar{U}$ (Mishra 1980) and σ are not linear and the coefficients of the governing Eqn. (1) cannot be expressed in the particular forms so as to get analytical solution of Eqn. (1). Therefore, we have to use centred difference scheme for vertical derivatives.

Consider perturbation stream function ψ' of the form

$$\psi'(x, p, t) = \psi_1(p, t)\cos kx + \psi_2(p, t)\sin kx \qquad (5)$$

Therefore, combining the system of $2N$ differential equations given by Eqn. (3) can be written as :

$$\frac{d}{dt}\begin{bmatrix} \psi_1^* \\ \vdots \\ \psi_N^* \\ \vdots \\ \psi_{2N}^* \end{bmatrix} = k.S \begin{bmatrix} \psi_1^* \\ \vdots \\ \psi_N^* \\ \vdots \\ \psi_{2N}^* \end{bmatrix} \tag{6}$$

where ψ_1^*, ψ_2^*, ..., ψ_N^*, ψ_{NH}^*, ..., ψ_{2N}^* are respectively, $\psi_1(1)$, $\psi_1(2)$, .., $\psi_1(N)$, $\psi_2(1)$, ..., $\psi_2(N)$ and the matrix S of the order $2N \times 2N$ is given by :

$$S = \left[\begin{array}{c|c} O & (B^{-1}A)_{N\times N} \\ \hline -(B^{-1}A)_{N\times N} & O \end{array}\right]_{2N\times 2N}$$

The system (6) can be integrated analytically w.r.t. time by assuming the solution of the form exp $(-kct)$, we get :

$$(S - c\,I)\,\psi^* = 0. \tag{7}$$

Once the eigenvalues c are known, the matrix of the eigen vectors can be formed and the solution of ψ^* can be obtained depending on initial value of ψ^* at each layer. Similarly the system (6) can be numerically integrated w.r.t. time and comparison of these two solutions of Eqn. (6) gives time truncation error; because vertical truncation error in both cases (due to centred difference scheme) is common. Further, in order to minimise time trucation error, numerical techniques can be used having more accuracy, *e.g.*, Bengtsson (1978) has used 4th order Rung-Kutta time integration scheme.

3.2. *Vertical truncation error*

We have seen that analytical solution of Eqn. (1) is not known. Therefore, as $\overline{U}=\overline{U}\,(p)$, $\sigma=\sigma\,(p)$ and $\psi\ (p,\,t_0)$ at initial time $t=t_0$ are known, vertical derivatives of $\overline{U}$, σ and ψ can be substituted analytically and the tendency $d\psi'/dt$ can be obtained. Further differentiate Eqn. (1) w. r. t. 't' and substitute for $d\psi'/dt$ as obtained from the previous procedure to get $d^2\psi'/dt^2$ and so on. Continuing this process higher order derivative of ψ' can be obtained. Now ψ' at next time step $t=t_1$ (say $t_1=t_0+\triangle t$) can be obtained using Taylor's expansion :

$$\psi'(t_1) = \psi'(t_0+\triangle t) = \psi'(t_0) + \triangle t\left(\frac{\partial\psi'}{\partial t}\right)_{t=t_0} + \frac{(\triangle t)^2}{2!}\left(\frac{\partial^2\psi'}{\partial t^2}\right)_{t=t_0} + \ \ldots\ldots$$

having sufficient number of terms so that error due to $\triangle t$ is negligible and the time derivatives at $t=t_0$ can be substituted from those obtained from the Eqn. (1) considering the equation in $\partial\psi'/\partial t$. Now ψ' obtained by this process contains no vertical as well as time truncation error. If this solution is compared with that of obtained from the system (6) analytically integrated w.r.t. time gives us vertical truncation error.

4. Truncation errors in phase velocity

Analytical solution of Eqn. (1) is possible only for the most simple case of constant basic parameters ($\overline{U}$=const., σ=const.). In this simple case considering $\psi' = \Psi\,(p)\,e^{ik\,(x-ct)}$, Eqn. (1) reduces to :

$$\frac{\partial^2\Psi}{\partial p^2} + q\,\Psi = 0 \tag{8}$$

where,

$$q = (\beta_0 - k^2\,\overline{U} + k^2\,c)\Big/(\overline{U}-c)\ \frac{f_0^2}{\sigma} \tag{9}$$

For $q>0$, solution of Eqn. (8) which satisfies the boundary condition (2) is given as $\Psi = a\cos\left(\sqrt{q}\,p\right)$

Therefore,

$$\sqrt{q}\,p_0 = n\,\pi,\ \ n=0,1,2,\ldots$$

Hence, the phase velocity c in the exact form can be obtained from (9) as :

$$c = \overline{U} - \frac{\beta_0/k^2}{1+n^2\,\alpha} \tag{10}$$

where,

$$\alpha = \frac{\pi^2 f_0^2}{p_0^2\,\sigma\,k^2} \tag{11}$$

which is same as obtained by Wiin Neilsen (1962). The solution for $q<0$ gives the phase speed same as obtained for $n=0$ in the previous case.

To evaluate vertical truncation error in the phase velocity, Eqn. (1) is to be integrated analytically w.r.t. time and numerically w.r.t. p using centred difference scheme and considering :

$$\psi'_j = A\cos\frac{n\pi\,\triangle p_j}{p_0}\,e^{ik\,(x-ct)}$$

suffix 'j' is for vertical grid point.

TABLE 1

Time truncation error (%)

n	1,500	5,000	10,000	1,500	5,000	10,000
	λ (km)					
	$\triangle t$=1 hr			$\triangle t$=1/2 hr		
1	.263	.0656	.0	.065	.016	.0
2	.279	.1093	.006	.069	.027	.002
3	.299	.1577	.026	.074	.039	.007
5	.33	.225	.05	.082	.056	.013
8	.3557	.269	.066	.088	.069	.016
10	.364	.282	.069	.090	.070	.017

TABLE 2

Vertical truncation error (%)

n	1,500	5,000	10,000	1,500	5,000	10,000
	λ(km)					
	N=5			N=10		
1	.02	.16	1.4	.01	.04	.35
2	.2	1.4	5.3	.06	.35	1.26
3	.8	3.5	8.4	.2	.8	1.9
5	2.7	8.7	15.0	0.6	1.7	2.6
8	7.7	26.11	56.2	1.27	2.8	3.4
10	10.78	43.1	181.2	1.76	3.74	3.75
	N=25			N=25		
1	.008	.01	.155	.0	.0	.056
2	.1	.156	.57	.0	.056	.2
3	.13	.375	.83	.0	.1	.297
5	.3	.74	1.07	.1	.26	.374
8	.5	1.07	1.27	.2	.36	.42
10	.6	1.25	1.415	.21	.4	.44

Then the phase velocity c ($=c_p$ say) takes the form :

$$c_p = \bar{U} - \frac{\beta_0 / k^2}{1 + \dfrac{2 f_0^2}{\sigma (\triangle p)^2 k^2} \left(1 - \cos \dfrac{n\pi \triangle p}{p_0} \right)}$$

or

$$c_p = \bar{U} - \frac{\beta_0 / k^2}{1 + \dfrac{4 f_0^2}{\sigma (\triangle p)^2 k^2} \sin^2 \left(\dfrac{n\pi \triangle p}{p_0} \right)}$$

Using sine series and further manipulation, c_p becomes,

$$c_p = c + (E_r)_p \qquad (12)$$

where,

$$(E_r)_p = \frac{\beta_0}{k^2 (1 + n^2 \alpha)} \left\{ \left[\frac{n^4 \pi^4 (\triangle p)^4 \alpha}{96 p_0^4} + ... \right] \times \right.$$

$$\left. \times (1 - n^2 \alpha + n^4 \alpha^2 - ...) \right\}$$

c is given by (10) and α is given by (11).

To find truncation error due to vertical and time derivatives, centred difference scheme for both the derivatives in Eqn. (1) is used considering :

$$\psi'_{j,m} = A \cos \frac{n\pi \triangle p_j}{p_0} \; e^{ik(x - cm \triangle t)}$$

suffix 'm' is for time grid point.

Then the phase velocity c ($=c_{pt}$ say) can be obtained as :

$$c_{pt} = \frac{1}{k.\triangle t} \sin^{-1} (k.\triangle t.c_p) \qquad (13)$$

where c_p is given by (12).

The truncation error due to time alone can be obtained using 'Leap-Frog' scheme for time derivatives and integrating the Eqn. (1) for constant basic parameters the phase velocity $c(=c_t$ say) can be written as :

$$c_t = \frac{1}{k.\triangle t} \sin^{-1} (k.\triangle t.c) \qquad (14)$$

where c is given by (10).

The expression (14) is same as (13) except c_p is replaced by c to eliminate vertical truncation error.

5. Results

Truncation errors in growth rate and the phase velocity of the baroclinically unstable short (λ=1500 km), intermediate (λ=5000 km) and long (λ=10000 km) waves embedded in the mean monsoonal zonal flow are odtained.

5.1. *Constant basic parameters*

In the previous study (MS) it is found that short unstable waves are essentially confined below 500 mb while the waves having wave-length $\geqslant$3500 km are above 500 mb. Therefore, the constant values of zonal wind $\overline{U}$ and inverse statistic stability σ^{-1} for mean monsoon situation are odtained, for the unstable

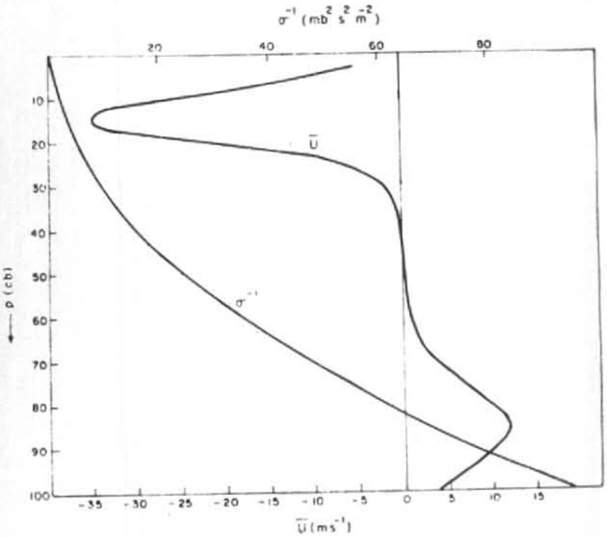

Fig. 1. Vertical profiles of mean monsoonal zonal wind $\bar{U}$ (ms^{-1}) and inverse static stability σ^{-1} ($mb^2\ s^2\ m^{-2}$)

TABLE 3

N	λ(km) 1,500	5,000	10,000
	(a) % error in growth rate		
15	14.796	5.3557	6.336
20	12.0	.684	1.3226
	(b) % error in phase velocity		
15	1.673	.6756	.5383
20	.386	.09	.08

waves, as their averaged values from the respective active region, and are given as :

For $\lambda = 1500$ km, $\bar{U} = 10\ ms^{-1}$, $\sigma^{-1} = 60\ mb^2\ s^2\ m^{-2}$

For $\lambda = 5000$ km & $= 10000$ km $\Big\}$ $\bar{U} = -30\ ms^{-1}$, $\sigma^{-1} = 12\ mb^2\ s^2\ m^{-2}$

Further, using expressions (10), (12) and (14) vertical truncation error $(c_p - c)$ and time truncation error $(c_t - c)$ in the phase velocity are computed choosing time step $(\triangle t)$ as 1 hr and 1/2 hr and varying number of layers (N) in vertical from 5 to 25. Percentage errors due to vertical and time truncation are presented in Tables 1 and 2 respectively. The error due to vertical truncation increases with increasing wave-length (λ), increasing cosine mode (n) and decreasing number of layers (N) in vertical. The time truncation error decreases with increasing wave-length (λ), decreasing cosine mode (n) and decreasing the time step $(\triangle t)$.

5.2. *Variable basic parameters*

The vertical profile of mean monsoonal zonal flow and inverse static stability presented in Fig. 1 are obtained analytically such that,

$$\bar{U} = U_W \ \mathrm{sech}^2 \left(\frac{p - P_W}{P_1} \right) - U_E \ \mathrm{sech}^2 \left(\frac{p - P_E}{P_2} \right)$$

and

$$\sigma^{-1} = a\,p^2 + b$$

where U_W, U_E and P_W, P_E are magnitudes and positions of westerly and easterly maximum wind respectively. The constants a, b, P_1 and P_2 are chosen such that the analytical profiles will have close resemblance with actual observed profiles. The mathematical representation of $\bar{U}$ used here fits more exact to the observed profile of $\bar{U}$ than that of used in MS.

For computation of growth rates and phase velocities, the expressions are used as given in MS. In this case the vertical truncation error is obtained by indirect method. Results for 25 layers in vertical are assumed to be closer to that of analytical solution. Also from the baroclinic instability study it was found that minimum 15 levels in vertical are required for the wave to be untable. Therefore, vertical layers (N) are chosen to be 15, 20 and 25. Tables 3(a) and 3(b) give percentage error in growth rate and phase velocity due to vertical truncation after comparing the results of 25 layers to less number of layers in vertical. The results show that vertical truncation error in phase velocity is much smaller than that in growth rates.

6. Conclusion

Simple expressions for quick estimation of vertical and time truncation errors in phase velocity, for the simple case of constant basic parameters using linearised quasi-geostrophic numerical model are obtained. The results show that time (vertical) truncation error decreases (increases) with increasing wave-length. The case of variable basic parameters show that the truncation error in growth rate is much larger than that in the phase velocity which indicates the importance of number of layers in vertical in the instability study.

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