A METHOD OF SOLUTION FOR TOPOGRAPHIC WAVES IN BAROCLINIC FLOW

For a two layer model on beta plane with the flow contained in a channel of width L, the dimensionless quasi-geostrophic equations (Pedlosky 1970, 1981) are :

$$\frac{\partial}{\partial t}[\nabla^2\Psi_n - \epsilon_n F(\Psi_1 - \Psi_2)] + J[\Psi_n, \nabla^2\Psi_n - \epsilon_n F(\Psi_1 - \Psi_2) + \beta y + \delta_n \eta] = 0 \quad (1)$$

where, $n=1, 2$; $\epsilon_1=1$, $\epsilon_2=-1$; $\delta_1=1$, $\delta_2=0$. Further, Ψ_1 and Ψ_2 are the non-dimensional geostrophic stream functions for layers 1 and 2 respectively. The parameter F is defined as :

$$F = (f_0^2 L^2/g) / (\triangle\rho/\rho) D$$

where, f_0 is the Coriolis parameter, g is gravity, $\triangle\rho/\rho$ is the fractional increase in density of the lower layer with respect to upper layer, D is the depth of each layer. The effects of Ekman layers, frictional forces and external vorticity sources are neglected. Further,

$$\eta = h_B/\epsilon D$$

where, $\epsilon = \overline{U}/f_0 L$ is the Rossby number. The topography is assumed to be sinusoidal,

$$\eta = \eta_0 \sin\theta \quad (2)$$

where,

$$\theta = kx + ly \quad (3)$$

k and l are characteristic parameters that yield configuration of bottom topography through Eqns. (2) and (3).

We will show that the governing Eqn. (1) may be solved exactly in steady state. We will look for a solution of the type :

$$\Psi_n = -U_n y + F_n \sin\theta \quad (4)$$

substitution of Eqn. (4) into Eqn. (1) yields :

$$F_1 = \eta_0 U_1 [(N^2 + F) U_2 - \beta_2] / R \quad 5(a)$$

$$F_2 = \eta_0 F U_1 U_2 / R \quad 5(b)$$

where,

$$R = [(N^2 + F) U_1 - \beta_1][(N^2 + F) U_2 - \beta_2] - F^2 U_1 U_2 \quad 6(a)$$

$$N^2 = k^2 + l^2 \quad 6(b)$$

$$\beta_n = \beta + \epsilon_n F (U_1 - U_2) \quad 6(c)$$

The resonance occurs when $R = 0$. in this case, the Doppler shifted wave field becomes stationary relative to the topography. We may rewrite Eqn. 6(a) as :

$$R = [(N^2 + 2F)\overline{U} - \beta](N^2 \overline{U} - \beta) - N^2(N^2 - 2F)S^2 \quad (7)$$

where,

$$\overline{U} = (U_1 + U_2)/2 \quad ; \quad S = (U_1 - U_2)/2 \quad (8)$$

If, the value of average zonal wind lies in an interval $[\beta/(N^2+2F), \beta/N^2]$ and if $N^2 < 2|F|$ for given value of F, then resonance will not occur for any chosen value of vertical shear.

The chosen solution (4) may also be recognized as a solution of the equation :

$$J[\Psi_n, \nabla^2\Psi_n + \beta_n y + \eta_n] = 0 \quad 9(a)$$

where,

$$\eta_n = [\delta_n \eta_0 - \epsilon_n (F_1 - F_2)] \sin\theta \quad 9(b)$$

Because, 9(a) is the steady state equation for a barotropic model in β_n plane, we conclude that the two-layer model in steady state with influence of topography in one of its layers is equivalent to two barotropic layers which are in steady state, with influence of topography, in β_1-plane and β_2-plane respectively.

Now, we will obtain a time dependent solution for sub-resonant flow be superimposition of disturbance stream functions ϕ_n on the topographic wave solution :

$$\Psi_n = -U_n y + F_n \sin\theta + \phi_n \qquad (10)$$

We look for a solution of the form :

$$\phi_n = \exp(i\lambda t)\sum_{m=-\infty}^{+\infty} P_{n,m}\exp(i\theta_m) + * \qquad (11)$$

where,

$$\theta_m = k_m x + l_m y \qquad 11(a)$$

$$(k_m, l_m) = (k_0, l_0) + m(k, l) \qquad 11(b)$$

and an * denotes complex conjugate. Substitution of (10) and (11) into (1) yield recursion relation in matrix form :

$$A_{m+1}\begin{pmatrix}P_{1,m+1}\\P_{2,m+1}\end{pmatrix} + B_m\begin{pmatrix}P_{1,m}\\P_{2,m}\end{pmatrix} + A_{m-1}\begin{pmatrix}P_{1,m-1}\\P_{2,m-1}\end{pmatrix} = 0 \qquad (12)$$

where,

$$A_m = \begin{pmatrix}F_1 d\, J_{1m}/U_1 & -FF_1 d\\ -FF_2 d & F_2 d\, J_{2m}/U_2\end{pmatrix} \qquad 13(a)$$

$$B_m = \begin{pmatrix}\lambda R_m^2 + k_m J_{1m} & -F(\lambda + U_1 k_m)\\ -F(\lambda + U_2 k_m) & \lambda R_m^2 + k_m J_{2m}\end{pmatrix} \qquad 13(b)$$

The parameters, used in 13(a,b), are :

$$d = (k\,l_0 - lk_0)/2 \qquad 14(a)$$

$$R_m^2 = k_m^2 + l_m^2 + F \qquad 14(b)$$

$$J_{nm} = U_n R_m^2 - \beta_n \qquad 14(c)$$

The recursion relation (12) can be used to determine Fourier coefficients $P_{n,m}$'s provided four coefficients are evaluated by use of boundary conditions, say known values of u, v at $x=0$, $y=0$, $t=0$ for each layer.

References

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