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Spectral model with a regional focus

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सार — सुनिश्चित स्थानीय मौसम प्रणाली, यदि वह सीमित सक्रिय क्षेत्र के बाहर वायुमंडलीय स्थिति के लिए अधिक संवेदी नहीं है, के अध्ययन के लिए चुनींदा उच्च विभेदन क्षेत्र के साथ भूमंडलीय मानावली निदर्श : उपयुक्त है। यदि अपेक्षित विभेदन का रैखिक आयाम हो और सक्रिय प्रदेश का क्षेत्र A हो तो हम पायेंगे कि एक पर्याप्त मानावली निदर्श में निम्नलिखित विशेषताएं हैं :

(क) मॉडल ध्रुव को भौगोलिक उत्तर से हटाकर अध्ययन के अधीन मौसम प्रणाली के परिवेश में पुर्नस्थापित किया जाता है।

(ख) नए गोलाकार प्रसंवादी y_{mn} के समुच्च आधारभूत कार्यों के रूप में उनके समानान्तर चर्तुभुजी खण्डित परास $|m| \leqslant n \leqslant N, 0 \leqslant |m| \leqslant M$ होने चाहिए जिनम $N \sim 2\pi\, a/l, M/N = \sqrt{4f(1-f)}, f = A/(4\pi a^2)$ और a पृथ्वी की त्रिज्या है।

चूंकि गोलीय निर्देशांक प्रणाली में सदिश क्षेत्रों के अपघटन में त्रिज्या और क्षैतिज भाग पूर्णत : निश्चर है, तो कोरिओलिस बल अवधि के अतिरिक्त गति के समीकरण का रूप मॉडल ध्रुव को पुर्नस्थापित करने से परिर्वातित नहीं होता। अरैखिक पदों के प्रबन्धन और खण्डन सहित मॉडलिंग के गणित में परिवर्तन नही होता है।

ABSTRACT. A global spectral model with selective high resolution region is appropriate for the study of a sharp local weather system if it is not very sensitive to the atmospheric state outside a limited active region. If the linear dimension of the resolution required is l and the area of active region is A then one finds that an adequate spectral model has the following features : (a) the model pole is moved away from the geographical north and re-located in the vicinity of the weather system under study, (b) the set of new spherical harmonics y_{mn} as basis functions should have the parallelogrammic truncated range $|m| \leqslant n \leqslant N$, $0 \leqslant |m| \leqslant M$, where $N \sim 2\pi\, a/l$, $M/N = \sqrt{4f(1-f)}$, $f = A/(4\pi a^2)$, where a is the radius of the earth.

Since the decomposition of a vector field, in a spherical coordinate system, into the radial and horizontal parts is rotationally invariant, the form of the equations of motion do not change under the re-location of the model pole except for the Coriolis force term. Mathematics of the modelling including truncation and management of non-linear terms do not change.

1. Introduction

If there exists a limited active region surrounding a local weather system such that the time evolution of the local weather system is not very sensitive to the atmospheric state outside the region then the weather system will be called fundamentally local (f.l.). It is conceivable that the non-linearity of atmospheric interactions permits the existence of f.l. systems. Limited area model (Krishnamurti 1969, Gadd 1984) would have been appropriate to study an f.l. system but for the problems with the boundary conditions. On the other hand the traditional global spectral models (Roberts 1966, Orszag 1970, Orszag 1974, Kasahara 1977, Machenhauer 1979, Jarraud and Simmons 1983, De Maria and Schubert 1984) with triangular truncation have isotropic resolution and hence to resolve the local feature one has to go to a model with the same high resolution everywhere. This means a large and expensive model. The variable mesh global models (Sharma *et al.* 1987) are beset with problems of anisotropy. Recently there has been an important advance in terms of conformal transform proposed by Schmidt (1977). It leads to a satisfactory spectral model with variable resolution without anisotropy in the horizontal. It has been applied using shallow water equations (Courtier and Geleyn 1988). This line of approach has to take up the challenge of strong convective systems of low latitudes and show that the vertical *vs.* horizontal anisotropy is inconsequential.

To attain regional high resolution one does not have to modify the spatial grid. We suggest utilizing the fact that for parallelogrammic truncation there is increased resolution at high latitudes at the expense of

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resolution in low latitudes (Machenhauer 1979). Appropriate truncation parameter for any desired resolution within a limited region around the north pole has been indicated in the next section. It is suggested that the model pole be relocated at the vicinity of f.l. system by an appropriate rotation from the traditional pole at the geographic north. The spherical harmonics associated with the model pole are expressible in terms of the traditional spherical harmonics through standard rotation matrices. An important property is that the spherical harmonics associated with the model pole provide a representation for the meteorological fields which is more stable near the f.l. system.

The shifting of the pole is effected rigorously through the appropriate representation of the rotation group. It follows that in the process no rotational property is vitiated; there is no seepage of anisotropy in the transformation process. The vertical direction is invariant under rotation and there does not develop, any new anisotropy in any vertical plane. In contradistinction the Schmidt transform (used by the ECMWF) uses a conformal transformation which does not have the nice properties of the rotation group and is bound to introduce new anisotropy in vertical planes.

The meridional low wave numbers that are inconsequential in the selected polar region can be used to improve the representation of the meteorological fields in the exterior region. They should ensure, in particular, that no prominent local system develops at the model 'south pole'.

2. Resolution and stability

Spectral model with triangular truncation is rotationally invariant and has uniform resolution all over the sphere. However, the stability of an expansion in spherical harmonics with respect to small errors in the determination of the expansion coefficients is maximal at the poles where a maximum number of spherical harmonics have common node.

For parallelogrammic truncation the resolution near the poles is higher than near the equator (Machenhauer 1979). For the polar region above a latitude φ_0 an east-west resolution of the order of a length l requires a range of m values $0 \leqslant |m| \leqslant M$, where $(2\pi a \cos \varphi_0)/l \sim M$. For the same order of north-south resolution we require a range of n values, $n \leqslant N$ where $(2\pi a)/l \sim N$. The area of the polar region is $2\pi(1-\sin\varphi_0)a^2$ and that is a fraction $f=(1/2)(1-\sin\varphi_0)$ of the global area. It follows that in a polar region of the fractional area $f=(\text{active area})/(4\pi a^2)$ a resolution length l can be maintained with the parallelogrammic truncation $N \sim 2\pi a/l$, $M/N = \cos\varphi_0 = 2\sqrt{f(1-f)}$. This model has a much coarser east-west resolution length $l/\cos\varphi_0$ at the equator although the north-south resolution remains unchanged. The number of spherical harmonics involved in the model is $(M+1)^2+(N-M)(2M+1)$.

To utilize the regional high resolution in the study of a f.l. system the pole of the spherical coordinate system has to be re-located in the vicinity of the f.l. system. This rotation of the coordinate system does not change the structure of the equations of motion except for the Coriolis force. It also does not vitiate any of the standard features contributing to the success of spectral model like no aliasing and applicability of the transform method for the management of the non-linear terms. This is a known observation and has been utilized by Schmidt (1977).

Stability of the spectral representation of the meteorological field in the vicinity of the pole is an important advantage. Consider, for example, a meteorological field $F(r, \lambda, \mu)$ expanded in terms of the spherical harmonics y_{mn} :

$$F = \sum_{m,n} C_{mn} Y_{mn} \qquad (1)$$

The maximal error $\triangle F$ induced in F due to errors $\triangle C_{m,n}$ in the coefficients is estimated to be :

$$\triangle F \sim \sum_{m,n} \left| \triangle C_{mn} \right| \left| Y_{mn} \right| \qquad (2)$$

which is small in the vicinity of the pole where all Y_{mn}, $m \neq 0$, have a common node.

It follows that the spectral representation of the meteorological fields in the active area of the f.l. system is more stable against errors in the modified model where the pole is located in the vicinity of the f.l. system than in the traditional model where f.l. system is away from the poles. In the latter case the local feature is built up through a delicate balance in spherical harmonics with large values. Slightest errors in the superposition coefficients can spoil the balance and wash off the local feature.

The meteorological fields are accurately represented, up to the resolution length l, within the active region. This leaves the super-position coefficients C_{mn} undetermined for $n-|m| < 2\pi/(\pi-2\varphi_0)$. These coefficients can be used to improve the representation of the field outside the active region. To ensure a clean single prominent f.l. system which is located around only the 'north' pole it is very important that the spherical harmonics with large (permitted) values of $n-|m|$ and $|m|$ do not superpose constructively at the 'south' pole. This can be achieved by imposing the following conditions :

$$\lim_{\mu \to -1} \sum_{n=|m|}^{N} C_{mn}(1-\mu^2)^{-m/2} P_{mn}(\mu) = 0 \text{ for each } m \qquad (3)$$

which may be utilized to determine C_{mm} and C_{-mm} for each m.

For the determination of the spectral coefficients by the least square deviation fit to the initial data we have to suitably smoothen the data outside the active region. Whereas the data within the active region should carry the full detail the data outside should be given on a sparse grid so that each grid point represents an average over an area as large as the active area.

3. Validity of the modification

There are two important features contributing to the success of spectral model, (a) consistency of the truncation of the inifinite series decomposition of the meteorological fields with the equations of motion, and (b) the transform method for handling the non-linear terms. To understand the genesis of these features we first note that the basic functions, which are the spherical harmonics, have a very specific transformation under rotations and that the equations of motion are composed of terms such that each transforms under rotation by an identical law. These two properties imply consistency of truncation. To establish the transform method one has to invoke, in addition, the property that for the set of spherical harmonics with orders constrained by an upper bound the integrals yielding their orthonormality relations can each be changed into sum over a fixed set of points. Now we notice that none of these properties are specific to the choice of axis or pole in defining the latitudes and longitudes. Thus, the fundamental properties that ensure success of traditional model also ensure the success of modified model with a shifted pole.

4. Methodology

We have to first establish the connection between the reference frame used in the traditional model and the reference frame used in the modified model. Subsequently we obtain the relationship between the spherical harmonics used in the traditional model and the spherical harmonics used in the modified model. Finally we obtain the relation between the expansion coefficients of the meteorological fields in the traditional model and the corresponding quantities in the modified model.

Let (α, β) be the latitude and longitude of the representative point of the region of interest. We define the Cartesian Model Frame of Reference (M) to have it's z-axis passing through the point (α, β). The traditional frame of reference, which we call as the Geographic Frame G has its z-axis coinciding with the axis of rotation of the earth. The Model Frame M is obtained from the Geographic Frame G through a sequence of rotations. For ready reference see, for example, Goldstein (1953) or Edmonds (1957). A rotation through β about the z-axis of the G-frame followed by a rotation through $(\pi/2-\alpha)$ about the intermediate y-axis and finally a rotation through $-\beta$ about the z-axis results in the M-frame.

Let the three components of a vector $\mathbf{V}$ in the G-frame be collectively denoted by a column of three numbers V^G and in the M-frame by the column of the three numbers V^M. The two columns are then related through the matrix representing rotation defined above.

$$V^M = R^C(-\beta, \alpha, \beta)\, V^G \tag{4}$$

$$R^C(-\beta, \alpha, \beta) = R_z(-\beta)\, R_y\left(\frac{\pi}{2}-\alpha\right) R_z(\beta),$$

$$R_z(\beta) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y\left(\frac{\pi}{2}-\alpha\right) = \begin{pmatrix} \sin\alpha & 0 & -\cos\alpha \\ 0 & 1 & 0 \\ \cos\alpha & 0 & \sin\alpha \end{pmatrix}$$

If the triad of the unit vectors in G are denoted by $\mathbf{e}(G)_i$ and in M are denoted by $\mathbf{e}(M)_i$ then :

$$[R^C(-\beta, \alpha, \beta)]_{ij} = \mathbf{e}(M)_i \cdot \mathbf{e}(G)_j.$$

Consider an arbitrary point (φ, λ). The triad of unit vectors of the Geographical Spherical Coordinate System G_s is denoted by $\mathbf{e}(G_s)_\lambda$, $\mathbf{e}(G_s)_\varphi$, $\mathbf{e}(G_s)_r$ which are eastward, northward and vertical directions respectively. This triad is obtained from the Cartesian triad $\mathbf{e}(G)_i$ through the following rotation :

$$R^P(\varphi, \lambda) = R_z\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{2}-\varphi\right) R_z(\lambda),$$

$$\{R^P(\varphi, \lambda)\}_{ij} = \mathbf{e}(G_s)_i \cdot \mathbf{e}(G)_j, \tag{5}$$

and hence the components V^{G_s} of the vector $\mathbf{V}$ in the geographical spherical coordinate system is given by

$$V^{G_s} = R^P(\varphi, \lambda)\, V^G \tag{6}$$

Consider the model spherical coordinate system M_s. Let (φ_M, λ_M) with respect to M_s denote the same point as (φ, λ) does with respect to G_s. Let V^{M_s} be the component of V in M_s. It follows that :

$$V^{M_s} = R^s(\varphi\lambda\,;\,\alpha\beta) V^{G_s} \tag{7}$$

where, $R^s(\varphi\lambda\,;\,\alpha\beta) = R^P(\varphi_M, \lambda_M)\, R^C(-\beta, \alpha, \beta)\, R^P(\varphi, \lambda)^{-1}$,

$$[R^s(\varphi\lambda\,;\,\alpha\beta)]_{ij} = \mathbf{e}(M_s)_i \cdot \mathbf{e}(G_s)_j.$$

Notice that the vertical axis or the z-axis in G_s and M_s are the same. Hence R^s represents a rotation about the common z-axis. Let γ be the angle of the rotation,

$$R^s(\varphi\lambda\,;\,\alpha\beta) = R_z\{\gamma(\varphi\lambda\,;\,\alpha\beta)\} \tag{8}$$

This determines γ in terms of $(\varphi\lambda)$ and $(\alpha\beta)$. The analytic expression for λ is lengthy but geometrical determination is direct. It is the angle between the geographical latitude and the present model latitude. Notice that any rotation R^C of the Cartesian coordinate system induces a rotation about the vertical at each point in space leaving the vertical itself invariant. Thus, decomposing a vector equation into vertical (radial) and horizontal is rotationally invariant. Horizontal components of V^{M_s} are expressed in terms of only the horizontal components of V^{G_s} and vertical component of V^{M_s} is identical with the vertical component of V^{G_s}.

Formal structure of the basic equations is not changed except for the Coriolis force. Since the components Ω^{G_s} of earth's angular velocity are Ω_0 $(0, \cos\varphi, \sin\varphi)$ it follows that :

$$\begin{aligned} \Omega^{M_s} &= R_z(\gamma)\, \Omega^{G_s} \\ &= \Omega_0 R^P(\varphi_M, \lambda_M)\, R^C(-\beta, \alpha, \beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{9}$$

$$\Omega_o = \frac{2\pi}{24\,\text{hr}},$$

$$i.e., \Omega^{M_s} = \Omega_o \begin{pmatrix} \sin\gamma \cos\varphi \\ \cos\gamma \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$= \Omega_o \begin{pmatrix} \cos\alpha \sin(\lambda_M - \beta) \\ \cos\alpha \sin\varphi_M \cos(\lambda_M - \beta) + \cos\varphi_M \sin\alpha \\ -\cos\alpha \cos\varphi_M \cos(\lambda_M - \beta) + \sin\varphi_M \sin\alpha \end{pmatrix} \quad (10)$$

The Coriolis force $-2 [\Omega \times \mathbf{V}]^{M_s}$ has now a somewhat lengthy expression. Decomposing Ω and $\mathbf{V}$ into vertical and horizontal parts :

$$-2(\vec{\Omega} \times \mathbf{V}) = -f\mathbf{k} \times \mathbf{V}_H - 2\vec{\Omega}_H \times \mathbf{k} V_3 - 2\vec{\Omega}_H \times \mathbf{V}_H \quad (11)$$

where,

$$\mathbf{V} = \mathbf{k} V_3 + \mathbf{V}_H, \; \vec{\Omega} = \frac{1}{2} f\mathbf{k} + \vec{\Omega}_H \quad (12)$$

$$f = 2\vec{\Omega}.\mathbf{k} = 2\Omega_o \; \frac{4\pi}{3} \sum_m y_{ml}^* \left(\pi + \beta, \; \frac{\pi}{2} - \alpha\right).$$

$$y_{ml} \left(\lambda_M, \; \frac{\pi}{2} - \varphi_M\right) \quad (13)$$

one can make the usual approximation,

$$-2(\Omega \times \mathbf{V}) \sim -f\mathbf{k} \times \mathbf{V}_H.$$

The form of the curvature terms do not change.

The connection between the normalized spherical harmonics $y_{m'n}(\lambda, \mu)$ and $y_{mn}(\lambda_M, \mu_M)$ is given by the $(2n+1) \times (2n+1)$ dimensional standard representation $D^{(n)}(R)$ of the rotation group :

$$y_{mn}(\lambda_M, \mu_M) = \sum_{n',m'} \delta\, n\, n' \, [D^{(n)}\{R^{C^{-1}}\}]_{m'm} \, y_{m'n'}(\lambda, \mu)$$

$$= \sum_m y_{m'n}(\lambda, \mu) \, [D^{(n)}\{R^{C^{-1}}\}]_{m'm} \quad (14)$$

where, $\mu = \sin\varphi$, $\mu_M = \sin\varphi_M$

The spherical harmonics expansion of any scalar field $\psi(r, \lambda, \mu)$ in the two frames read as :

$$\psi(r, \lambda, \mu) = \sum_{nm} \psi^G_{nm} \, y_{mn}(\lambda, \mu) \quad (15)$$

$$= \sum_{nm} \psi^M_{mn} \, y_{mn}(\lambda_M, \mu_M).$$

The coefficients are seen to be related as :

$$\psi^M_{mn} = \sum_{m'} D^{(n)}_{mm'}(R_C) \, \psi^G_{m'n} \quad (16)$$

where we notice that the equality involves a single value for n, *i.e.*, it does not mix up different n's.

It follows that the management of parallelogrammic truncation in the model frame M_s is identical to that in the conventional frame G_s.

Let us now consider a vector field, for example, the velocity field. Let $C^{G_s\,n}_{\alpha\,m}$ and $C^{M_s\,n}_{\alpha\,m}$ be the spectral expansion coefficients in the geographical and modified spherical coordinate systems respectively :

$$V^{G_s}_\alpha = \sum_{nm} C^{G_s\,n}_{\alpha\,m} \, y_{mn}(\lambda, \mu),$$

$$V^{M_s}_\alpha = \sum_{nm} C^{M_s\,n}_{\alpha\,m} \, y_{mn}(\lambda_M, \mu_M)$$

In view of the Eqns. (7), and (8) and (14) it follows that

$$C^{M_s\,n}_{\alpha\,m} = \sum_{\beta m'} [R_z(\gamma)]_{\alpha\beta} \, D^{(n)}_{mm'}(R_c) \, C^{G_s\,n}_{\beta\,m'} \quad (17)$$

where it should be noted that for $\alpha = 3$ it reads :

$$C^{M_s\,n}_{3\,m} = \sum_{m'} D^{(n)}_{mm'}(R_c) \, C^{G_s\,n}_{3\,m'} \quad (18)$$

Thus, the coefficient of the vertical component does not mix with the coefficients of the horizontal components under the transformation from G_s to M_s. Further, there is no mixing of different n's—which means no new problems of truncation.

4. Conclusion

The fundamental advantages of the traditional spectral model are not lost as the pole of the basis functions is shifted from the geographical pole to the region of interest. One gains the advantage of a selective high resolution look and a stable representation in the region of interest through the parallelogrammic truncation. The maximum values of n and $|m|$ is determined by the needed resolution length and the area of active region. The degrees of freedom in the model are closely fitting to the regional needs.

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Appendix

Spectral decomposition of product of fields can be expressed in terms of Clebsch Gordon coefficients. There may be advantage to such expressions because Clebsch Gordon coefficients have been extensively studied and tabulated. Basic equations in this context relate (a) product of spherical harmonics as linear superposition of spherical harmonics and (b) derivatives of spherical harmonics as linear superposition of spherical harmonics.

$$y_{m_1 n_1}(\lambda, \mu)\, y_{m_2 n_2}(\lambda, \mu) = \sum_{nm} (yy)^{n_1\ n_2\ n}_{m_1\ m_2\ m}\, y_{mn}(\lambda, \mu)\,, \tag{A1}$$

$$(yy)^{n_1\ n_2\ n_3}_{m_1\ m_2\ m_3} = \left[\frac{(2n_1+1)\,(2n_2+1)}{4\pi\,(2n_3+1)}\right]^{1/2} C^{n_1\ n_2\ n_3}_{0\ 0\ 0}\; C^{n_1\ n_2\ n_3}_{m_1\ m_2\ m_3} \tag{A2}$$

where, $C^{n_1\ n_2\ n_3}_{m_1\ m_2\ m_3}$ are the Clebsch Gordon coefficients; $\mu = \sin\varphi$.

$$\frac{\partial}{\partial\varphi} y_{mn}\left(\lambda, \frac{\pi}{2}-\varphi\right) = \overset{(+)}{D}_{2,m,n} Y_{m+1,n}\left(\lambda, \frac{\pi}{2}-\varphi\right) + \overset{(-)}{D}_{2,m,n} Y_{m-1,n}\left(\lambda, \frac{\pi}{2}-\varphi\right) \tag{A3}$$

where,

$$\overset{(+)}{D}_{2,m,n} = -\frac{e^{-i\lambda}}{2}\{(n-m)\,(n+m+1)\}^{1/2}$$

$$\overset{(-)}{D}_{2,m,n} = -(\overset{(+)}{D}_{2,-m,n})^{*} \tag{A4}$$

$$\frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda} y_{mn}\left(\lambda, \frac{\pi}{2}-\varphi\right) = \overset{(+)}{D}_{1,m,n} Y_{m+1,n+1}\left(\lambda, \frac{\pi}{2}-\varphi\right) + \overset{(-)}{D}_{1,m,n} Y_{m-1,n+1}\left(\lambda, \frac{\pi}{2}-\varphi\right) \tag{A5}$$

where,

$$\overset{(+)}{D}_{1,m,n} = -\frac{i}{2}\left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2} e^{-i\lambda}$$

$$\overset{(-)}{D}_{1,m,n} = -\frac{i}{2}\left[\frac{2n+1}{2n+3}\,\frac{n-m+2}{n-m+1}\right]^{1/2}(n+3m)\, e^{i\lambda} \tag{A6}$$

under the assumption $m>0$.

$$\text{Let } \psi(r, \varphi_M, \lambda_M) = \sum_{m\,n} \overset{M}{\psi}_{mn}(r)\, y_{mn}\left(\lambda_M, \frac{\pi}{2} - \varphi_M\right),$$

then product with the Coriolis parameter f which depends on φ_M, λ_M leads to the expansion:

$$f(\varphi_M, \lambda_m)\,\psi(r,\varphi_M,\lambda_M) = 2\Omega_0 \frac{4n}{3} \sum_{\substack{n,m\\ n',m',m''}} Y^*{}_{m''l}\left(\pi+\beta, \frac{\pi}{2}-\alpha\right) \overset{M}{\psi}_{m'n'}(r).$$

$$(yy)^{l'\;\;n'\;\;n}_{m''\;m'\;m} \cdot Y_{mn}\left(\lambda_M, \frac{\pi}{2} - \varphi_M\right) \tag{A7}$$

The often occurring Jacobian has the following decomposition :

$$F(\psi) = \frac{a^2}{\cos\varphi}\left(\frac{\partial\psi}{\partial\varphi}\,\frac{\partial}{\partial\lambda}\nabla^2\psi - \frac{\partial\psi}{\partial\lambda}\,\frac{\partial}{\partial\varphi}\nabla^2\psi\right)$$

$$= \sum_{\substack{n_1, n_2, n_3\\ m_1, m_2, m_3}} \left[n_2(n_2+1) - n_1(n_1+1)\right] \overset{M}{\psi}_{m_1n_1}\, \overset{M}{\psi}_{m_2n_2}.$$

$$\Big\{\overset{(+)}{D}_{2,m_1,n_1}\overset{(-)}{D}_{1,m_2,n_2}(yy)^{n_1\;\;n_2+1\;\;n_3}_{m_1+1\;m_2-1\;m_3}$$

$$+\overset{(+)}{D}_{2,m_1,n_1}\overset{(+1)}{D}_{1,m_2,n_2}(yy)^{n_1\;\;n_2+1\;\;n_3}_{m_1+1\;m_2+1\;m_3}$$

$$+\overset{(-)}{D}_{2,m_1,n_1}\overset{(-)}{D}_{1,m_2,n_2}(yy)^{n_1\;\;n_2+1\;\;n_3}_{m_1-1\;m_2-1\;m_3}$$

$$+\overset{(-)}{D}_{2,m_1,n_1}\overset{(+)}{D}_{1,m_2,n_2}(yy)^{n_1\;\;n_2+1\;\;n_3}_{m_1-1\;m_2+1\;m_3}\Big\}\, Y_{m_3n_3}$$