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Study of storm surge problem on a continental shelf with sloping bottom

K. HALDAR

Electronics Unit

Indian Statistical Institute, Calcutta

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सार — वर्तमान अन्वेषण महाद्वीपीय कगार (शेल्फ) पर तूफान महोर्मि समस्या का सैद्धांतिक अध्ययन प्रस्तुत करता है। गहरे समुद्र से उत्पन्न तूफान आदि हलचल के कारण तटीय तल पर यह समस्या देखी जाती है। वर्तमान समस्या का अध्ययन करते हुए यह देखा गया है कि तटीय ज्यामिति रचना में एक ऊर्ध्वाधर सीधी दीवार होती है जो ढलवां तल वाली क्षैतिज दिशा में अनन्त की ओर बढ़ती चली जाती है। इस विश्लेषण में समीकरण प्रणाली के रैखिक निदर्श पर विचार किया गया है, साथ ही कोरिआलिस प्राचल एवं तलीय घर्षण के प्रभावों पर भी विचार किया गया है। महाद्वीपीय तट पर समुद्र सतह उन्नयन (उठान) के लिए विश्लेषणात्मक पद, इस सैद्धांतिक अन्वेषण से प्राप्त मुख्य परिणाम हैं।

ABSTRACT. The present investigation represents the theoretical study of storm surge problem on a continental shelf. This problem is found to happen on the coastal bed due to a disturbance, such as storm, generated in the deep sea. In studying the present problem the coastal geometry consists of a vertical straight wall extending to infinity in horizontal direction with sloping bottom. In this analysis the linear model of the system of equations has been considered; the effects of coriolis parameter and bottom friction have also been taken into account. The main result obtained in this theoretical investigation is the analytical expression for the sea surface elevation on the continental bed.

1. Introduction

Storm surge generation is a complex phenomenon and it happens due to interaction of air/sea and sea/land separations concerning lower atmosphere. It has devastating effect on the properties and lives of human beings in the coastal regions.

Many works based on the storm surges have been done and some of them which are relevant to the present investigation, have also been reviewed here. Heaps (1965) considered a problem which incorporated a continental shelf of uniform depth and finite width bounded by a long straight coast and connected with and infinite deep ocean. Das *et al.* (1974), using the concept of Heaps (1969) treated a problem on storm surge which affected the coast of Bangladesh during November 1970 and calculated the elevation of sea surface at the coast.

Ghosh (1977) on the basis of Jelesnianski's (1967) works, evolved an objective technique of prediction of storm on the east coast of India. Storm surge problems in the Bay of Bengal have also been studied by Das (1972), Johns and Ali (1980), Johns *et al.* (1981, 1982, 1983) and Ghosh (1982).

The present investigation, using the idea of Heaps (1965), gives the technique of solution of governing equations in respect of a sloping bottom with a hypothetical surge input along its oceanic edge. In this analysis the following assumptions are made :

(1) The coastal bed consists of a vertical straight wall extending to infinity in horizontal direction with sloping bottom. It has finite width and is connnected with an infinitely deep ocean.

(2) The velocity components and the vertical displacement are small, therefore, the terms which involve their squares and products are discarded from the equations of motion.

(3) The vertical acceleration is negligible and the horizontal motion for all particles on the continental shelf is the same in the same vertical line.

(4) The motion on the continental bed is the same for all sections normal to the coast, as a result of which the dependent variables are all independent of y.

(5) The atmospheric pressure over the entire bed is uniform.

(6) The effects of coriolis force and bottom friction are taken into account whereas the effects of air/sea interaction on the bed is neglected.

(7) The bottom friction obeys the linear law.

(8) The input of the surge at the oceanic edge due to a storm generated in the deep sea is represented by the form of a half sine wave with time variation.

2. Mathematical formulation

A rectangular cartesian coordinate system is used, in which the origin is taken on the undisturbed surface of the sea and located at a point on the coast. In this system the x-axis is normal to the coast, the y-axis is parallel to it and the z-axis is directed vertically downwards. The coastal bed consists of a vertical straight wall extending to infinity on both sides with a sloping bottom described by $H(x) = H_0 + \gamma x$, where γ is the slope of the bed. If η be the displacement of the disturbed surface of the sea above the equilibrium level and if u, v be the horizontal components of the velocity parallel to x and y, then the linear system of the dynamic equations is

$$\begin{aligned} \frac{\partial}{\partial x}(UH) + \frac{\partial \eta}{\partial t} &= 0 \\ \frac{\partial U}{\partial t} - fV + g\frac{\partial \eta}{\partial x} &= -\frac{F_B}{\rho H} \\ \frac{\partial V}{\partial t} + fU &= -\frac{G_B}{\rho H} \end{aligned} \tag{1}$$

where $f = 2\Omega \sin \varphi$ is the coriolis parameter assumed to be constant and Ω is the angular velocity of the rotating earth at a latitude φ. Here F_B, G_B are the xy-components of the frictional stress at the bottom, and U, V are the depth-mean values of u, v defined by

$$\begin{aligned} U &= \frac{1}{H+\eta}\int_H^{-\eta} u\,dz \\ V &= \frac{1}{H+\eta}\int_H^{-\eta} v\,dz \end{aligned} \tag{2}$$

Applying the linear law given by :

$$F_B = 2k\rho U$$
$$G_B = 2k\rho V$$

one obtains the system of equations as

$$\begin{aligned} \frac{\partial}{\partial x}(UH) + \frac{\partial \eta}{\partial t} &= 0 \\ \frac{\partial U}{\partial t} + 2kU - fV + g\frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial V}{\partial t} + 2kV + fU &= 0 \end{aligned} \tag{3}$$

where k is some constant.

The initial conditions are

$$U = V = \eta = 0 \quad \text{when } t = 0 \tag{4}$$

3. Determination of the sea-surface elevation on the continental shelf

In this section the response of sea-surface on the continental bed will be determined and it is assumed that the part of the surge is due to disturbances generated in the deep sea and propagated on to the shelf. For determining the elevation of sea-surface on the bed, the input of surge at the oceanic edge is given by

$$\eta = \eta_m \sin \omega t \quad \text{at } x = l \tag{5}$$

where l is the width of bed. The relation (5) shows that the elevation of the sea surface at the oceanic edge varies with time, *i.e.*, it rises from zero at $t = 0$ and executes a simple harmonic oscillation of amplitude η_m and period $2\pi/\omega$

Taking Laplace transforms in (3) and applying the initial conditions (4) one obtains

$$\frac{d}{dx}(\bar{U}H) + \delta\,\bar{\eta} = 0 \tag{6}$$

$$(2k + S)\,\bar{U} - f\bar{V} + g\frac{d\bar{\eta}}{dx} = 0 \tag{7}$$

$$(2k + S)\,\bar{V} + f\bar{U} = 0 \tag{8}$$

where the Laplace transform of the function $Q(x, t)$ is defined by :

$$\bar{Q}(x, S) = \int_0^{a} e^{-st}\, Q(x, t)\, dt \tag{9}$$

Solving (7) and (8) for U, one gets

$$\bar{U} = -\frac{g(2k+S)}{(2k+S)^2 + f^2} \cdot \frac{d\bar{\eta}}{dx} \tag{10}$$

which reduces the Eqn. (6) to the following form

$$(H_0 + \gamma x)\frac{d^2\bar{\eta}}{dx^2} + \gamma\frac{d\bar{\eta}}{dx} - \beta^2\bar{\eta} = 0 \tag{11}$$

where,

$$\beta^2 = \frac{S\{(2k+S)^2 + f^2\}}{g(2k+S)} \tag{12}$$

$$H(x) = H_0 + \gamma x \tag{13}$$

and γ is the slope of the bed.

The solution of the Eqn. (11) is

$$\bar{\eta} = A\,I_0\left(\frac{2\beta}{\gamma}\sqrt{H_0 + \gamma x}\right) + B\,K_0\left(\frac{2\beta}{\gamma}\sqrt{H_0 + \gamma x}\right) \tag{14}$$

where I_0, K_0 are modified Bessels' functions of order zero and A, B are arbitrary constants to be determined from the boundary conditions, one of which is given by (5) and the other is

$$u = 0 \quad \text{at} \quad x = 0 \tag{15}$$

The Laplace transforms of these boundary conditions are

$$\begin{aligned} \bar{\eta} &= \eta_m\,\omega/(\delta^2 + \omega^2) \quad \text{at} \quad x = l \\ \bar{U} &= 0 \quad \text{at} \quad x = 0 \end{aligned} \tag{16}$$

Applications of the boundary conditions (16) give

$$A = \eta_m \frac{\omega}{S^2+\omega^2} \cdot \frac{K_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right)}{K_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) I_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right) + I_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) K_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right)} \tag{17}$$

$$B = \eta_m \frac{\omega}{S^2+\omega^2} \cdot \frac{I_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right)}{K_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) I_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right) + I_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) K_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right)}$$

and the expression for $\bar{\eta}$ becomes

$$\bar{\eta}(x, S) = \eta_m \frac{\omega}{S^2+\omega^2} \cdot \frac{K_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) I_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma x}\right) + I_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) K_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma x}\right)}{K_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) I_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right) + I_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) K_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right)} \tag{18}$$

Hence the inverse theorem of Laplace transformation gives

$$\eta(x, t) = \frac{1}{2\pi i} \int_{\delta - ia}^{\delta + ia} e^{st}\, \bar{\eta}(x, S)\, dS \tag{19}$$

where $\bar{\eta}(x, S)$ is given by (18), and δ is real and positive.

The next task is to evaluate the integral (19) using Cauchy's residue theorem. The details of evaluation of the integral are given in Heaps (1965) and Thomson (1962). The integrand has the singularities at $S = \pm i\omega$ and at roots of

$$K_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) I_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right) + I_1\left(\frac{2\beta}{\gamma}\sqrt{H_0}\right) K_0\left(\frac{2\beta}{\gamma}\sqrt{H_0+\gamma l}\right) = 0 \tag{20}$$

which can be written as

$$K_1(\alpha_n \sqrt{H_0})\, I_0(\alpha_n \sqrt{H_0+\gamma l}) + I_1(\alpha_n \sqrt{H_0})\, K_0(\alpha_n \sqrt{H_0+\gamma l}) \tag{21}$$

where, $\quad \beta = \dfrac{\gamma\, \alpha_n}{2}$ (22)

from which, using (12), one gets

$$S\left\{(2k+S) + \frac{f^2}{2k+S}\right\} = \frac{g\,\gamma^2\,\alpha_n^2}{4} \tag{23}$$

The Eqn. (23) is a cubic equation is S and when

$$\frac{16k^2}{3} < 4f^2 - g\,\gamma^2\,\alpha_n^2 \tag{24}$$

the roots are of the form

$$-\lambda_n,\ -\mu_n \pm i\, v_n \tag{25}$$

and λ_n, μ_n, v_n, are given by the expressions

$$\begin{aligned} \lambda_n &= \frac{4k}{3} + \frac{1}{\sqrt[3]{2}}\left\{(G - \sqrt{G^2+4H^3})^{1/3} + (G + \sqrt{G^2+4H^3})^{1/3}\right\} \\ \mu_n &= 2k - \frac{1}{2}\lambda_n\ , \quad v_n = \frac{1}{4}\left(12\,\lambda_n^2 - 32\,k\,\lambda_n + 16f^2 - g\,\gamma^2\,\alpha_n^2\right)^{1/2} \end{aligned} \tag{26}$$

where H and G are of the following forms

$$H = \frac{1}{36}\left\{3\,(4f^2 - g\,\gamma^2\,\alpha_n^2) - 16k^2\right\},\ G = -\frac{k}{54}\left\{9\,(8f^2 + g\,\gamma^2\,\alpha_n^2) + 32k\right\} \tag{27}$$

Therefore, the resulting expression for the sea surface elevation on the continental shelf is

$$\begin{aligned} \frac{\eta}{\eta_m} = \sum_{n=1}^{a} \Bigg[& \frac{2\omega\, e^{-\mu_n t}\{K_1(\alpha_n\sqrt{H_0})\, I_0(\alpha_n\sqrt{H_0+\gamma x}) + I_1(\alpha_n\sqrt{H_0})\, K_0(\alpha_n\sqrt{H_0+\gamma x})\}\,(L_n \cos v_n t + M_n \sin v_n t)}{E_n(L_n^2 + M_n^2)} \\ & + \frac{\omega\, g\,\gamma\,\alpha_n (2k-\lambda_n)^2\{K_1(\alpha_n\sqrt{H_0})\, I_0(\alpha_n\sqrt{H_0+\gamma x}) + I_1(\alpha_n\sqrt{H_0})\, K_0(\alpha_n\sqrt{H_0+\gamma x})\}\, e^{-\lambda_n t}}{2\,E_n(\omega^2+\lambda_n^2)\{(k-\lambda_n)(2k-\lambda_n)^2 + kf^2\}} \Bigg] + \\ & + \frac{(C_l D_x - C_x D_l)\cos\omega t + (C_x C_l + D_x D_l)\sin\omega t}{C_l^2 + D_l^2} \end{aligned} \tag{28}$$

where L_n, M_n, E_n, C_l, D_l, C_x and D_x, being represented by large expressions, are not given here. These expressions can be obtained by straight forward calculations and are given in Haldar (1983).

The expression (28) gives the response of sea surface on the continental shelf due to motion generated at the oceanic edge by an oscillation (5). Therefore, it follows that for an oceanic surge incident on the shelf and represented by the rise and fall of water at the oceanic edge given by the half wave

$$\begin{aligned} \eta &= \eta_m \sin \omega t \qquad (0 \leqslant t \leqslant \pi/\omega) \\ &= 0 \qquad (t \geqslant \pi/\omega) \end{aligned} \tag{29}$$

the elevation of the sea surface on the continental bed is

$$\begin{aligned} \eta &= \eta(x, t) & (0 \leqslant t \leqslant \pi/\omega) \\ \eta &= \eta(x, t) + \eta(x, t - \pi/\omega) & (t \geqslant \pi/\omega) \end{aligned} \tag{30}$$

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