

Experiments with the balance equation

M. C. SHARMA and M. C. SINHA

Meteorological Office, Poona

ABSTRACT. Numerical experiments were performed to solve the balance equation for obtaining the stream function from the geopotential field. The balance-wind derived from the balance- ψ is then compared with geostrophic wind. It is observed that the computed balance-wind appears to be a better approximation to the observed wind field than the geostrophic wind derived from the observed geopotential.

1. Introduction

In numerical forecast one needs a wind for the advection of absolute vorticity. This advective wind could be obtained either through stream function computed from the direct analysis of the wind field or with the help of geostrophic assumption or through the balance equation.

Several workers (Bolin 1955, 1956; Charney 1955; Shuman 1957; Miyakoda 1956, 1960) have suggested the use of balance wind derived from balance equation rather than the use of geostrophic wind for prognostic models.

Ramanathan *et al.* (1971) solved the reverse balance equation to compute the wind vector values at grid points and then compared the geopotential field from the analysis of the wind field over the Indian region. In this diagnostic study the authors have taken the geopotential field as the basic input and have solved the balance equation for the stream function field. The objective is to find out whether such a derived stream function is able to represent the observed wind field so as to use this subsequently in a prognostic model.

2. Balance equation

Charney (1955) has expressed the most general relation between pressure (contour heights or geopotential) and winds in the form of the balance equation, which is a special form of the divergence equation. It may be written as,

$$2J(u, v) + \frac{\partial}{\partial x}(fv) - \frac{\partial}{\partial y}(fu) = \nabla^2\phi \quad (1)$$

where,

- ϕ is the geopotential
- u is the zonal component of the wind
- v is the meridional component of the wind

f is the coriolis parameter and

J is the Jacobian operator.

If ψ is defined to be the stream function, then the non-divergent velocity components will be given by

$$u = -\frac{\partial\psi}{\partial y} \text{ and } v = \frac{\partial\psi}{\partial x} \quad (2)$$

Insertion of (2) into (1) leads to the so called Monge-Ampere equation as,

$$2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) + \frac{\partial}{\partial x}(f\psi_x) + \frac{\partial}{\partial y}(f\psi_y) = \nabla^2\phi \quad (3)$$

where,

$$\psi_x = \frac{\partial\psi}{\partial x}, \psi_{xx} = \frac{\partial^2\psi}{\partial x^2}, \psi_{xy} = \frac{\partial^2\psi}{\partial x\partial y} \text{ etc.} \quad (4)$$

If $A = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ and $B = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$

are shearing stresses and deformations (Petterssen 1956) respectively, then they may also be written in terms of partial derivatives of ψ as,

$$A = \psi_{xx} - \psi_{yy} \text{ and } B = -2\psi_{xy}$$

Introducing these relations into (3) we have

$$\frac{1}{2} [(\nabla^2\psi)^2 - (A^2 + B^2)] + f\nabla^2\psi + f_x\psi_x + f_y\psi_y = \nabla^2\phi$$

and solving for $\nabla^2\psi$, we have

$$\nabla^2\psi = -f \pm [2\nabla^2\phi + f^2 + (A^2 + B^2) - 2(f_x\psi_x + f_y\psi_y)]^{1/2}$$

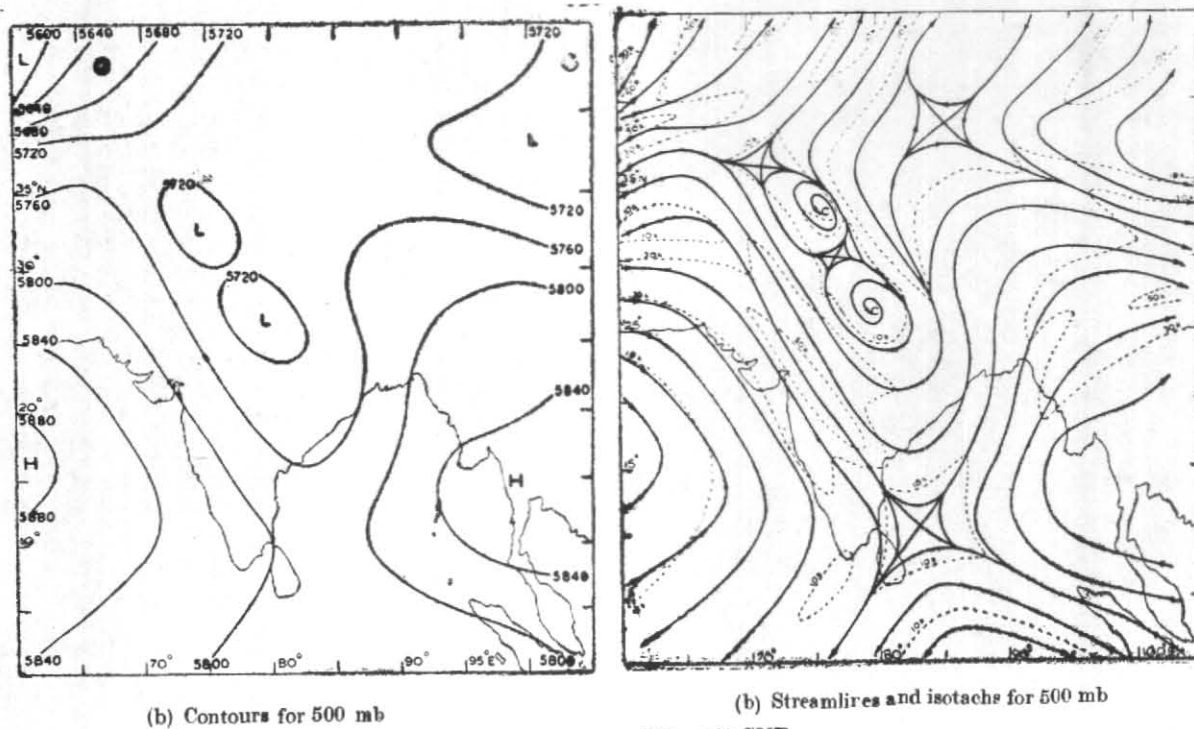


Fig. 1. Case I—19 May 1971 at 00 GMT

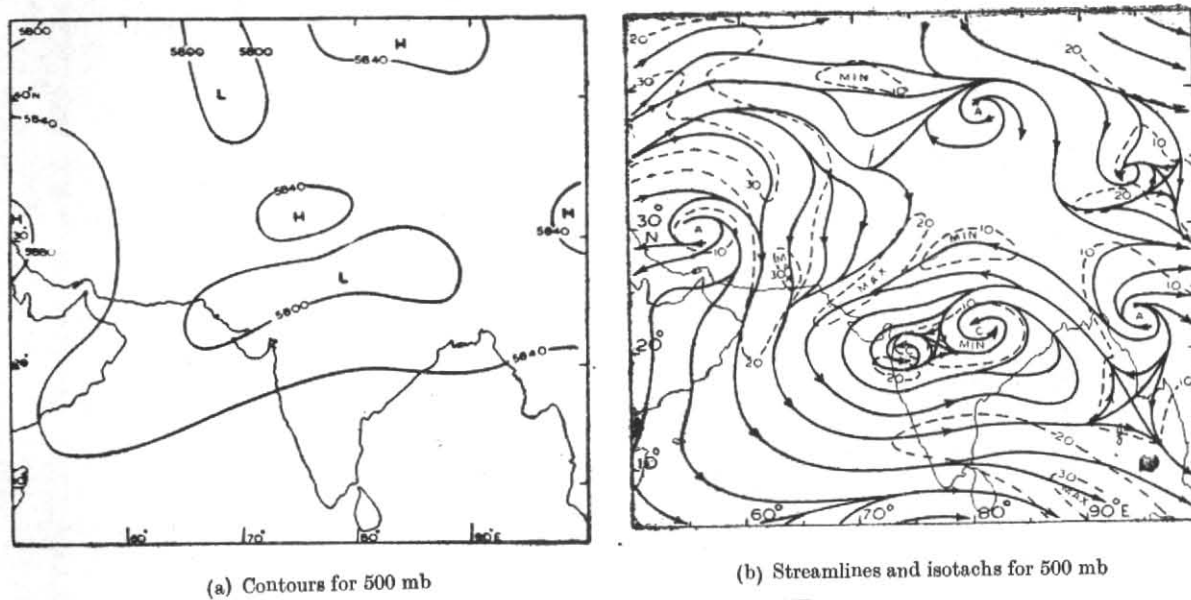


Fig. 2. Case III—7 July 1963 at 1200 GMT

So far as the large scale disturbance is concerned, the solution of positive sign should be taken in the northern hemisphere and that of negative sign in southern hemisphere (Miyakoda 1960)

$$\nabla^2\psi = -f + [2\nabla^2\phi + f^2 + (A^2 + B^2) - 2(f_x\psi_x + f_y\psi_y)]^{1/2}$$

Putting $C = 2(f_x\psi_x + f_y\psi_y)$,

$$\text{then } \nabla^2\psi = -f + [2\nabla^2\phi + f^2 + (A^2 + B^2) - C]^{1/2} \quad (5)$$

for real values of ψ , we must have

$$2\nabla^2\phi + f^2 + (A^2 + B^2) - C > 0 \quad (6)$$

3. Data and region of study

We have studied two synoptic situation—Cases I and II. In Case I the data used was of 500 mb on 19 May 1971 (00 GMT). The region of study was from 2° to 44°N and 60° to 102°E. The contour analysis and the streamline isotach analysis are given in Figs. 1(a) and 1(b) respectively. In Case II the data used was of 500 mb on 7 July 1963 (12 GMT). The region of study was from 7.5° to 45°N and 50° to 100°E. The contour analysis and the streamline isotach analysis are given in Figs. 2(a) and 2(b) respectively.

4. Method of solving the balance equation

Various techniques of solving balance equation are suggested by different workers. Mention may be made of Shuman (1957), Bolin (1956), Miyakoda (1956, 1960), Bushby and Huckle (1956), Bring and Charasch (1958) etc. Here we have mainly followed the method 'C' of Miyakoda (1960), which uses the finite differencing scheme as suggested by Bring and Charasch (1958).

4.1. Ellipticity condition

Eq. (3) is the Monge-Ampère equation and can be solved under the condition of ellipticity (Bolin 1956) given by,

$$\frac{\nabla^2\phi}{f} - \frac{1}{f} \nabla f \cdot \nabla \psi > -\frac{f}{2}$$

which may also be written as,

$$2\nabla^2\phi + f^2 - 2(f_x\psi_x + f_y\psi_y) > 0$$

The corresponding condition for the Eq. (5) is,

$$2\nabla^2\phi + f^2 - C > 0 \quad (7)$$

From the Eq. (5), a restriction on ψ is necessary, *i.e.*, the radicand must not be negative. Since $(A^2 + B^2)$ in Eq. (5) is necessarily positive $(2\nabla^2\phi + f^2 - C)$ must also be positive for ψ to be real which is incidentally also the condition of ellipticity of the balance equation,

In the actual atmosphere, the condition (7) may not be satisfied around the anticyclones. In those circumstances Shuman (1957) suggested to modify the ϕ field so that the criterion (6) of ellipticity may be satisfied. Such a procedure would be required for those points where the first two terms of Eq. (7) are small and C is sufficiently positive to violate the condition (7) which in turn makes the radicand (6) negative. This can be handled by substituting zero for the radicand in (6) where the condition is violated. By enforcing this condition, ψ can be solved from (5) for real values by relaxation methods. In the course of our computations, the ellipticity condition failed at about 12 per cent points in Case I and at no points in Case II. In the Case I the condition failed in the anticyclonic region of streamlines (Miyakoda 1960). In the Case II, since the flow was dominated by the monsoon trough and anticyclones were absent in the field, the ellipticity condition was not violated in the initial data itself.

4.2. Differencing scheme for determining $(A^2 + B^2)$ and C .

Four methods were discussed by Miyakoda (1960) to determine $(A^2 + B^2)$ and C . He has recommended the method C to determine $(A^2 + B^2)$ based on rotated mesh through 45°, as in Fig. 3, because $(A^2 + B^2)$ is invariant (Pettersen 1956) with respect to the rotation of coordinates and also satisfies the Gaussian theorem. Since the Laplacian is also invariant, all the terms in $(A^2 + B^2)$ can be calculated with the rotated mesh.

Expressing A and B in terms of partial derivatives of ψ , as defined by (4) after some manipulation of the terms we may write,

$$A^2 + B^2 = (\nabla^2\psi)^2 - 4[\psi_{xx}\psi_{yy} - \psi_{xy}^2]$$

If d is the grid length, α the latitude and $d \cos \alpha$ the latitude correction, $(A^2 + B^2)$ can be written in the difference form with reference to Fig. 3 as,

$$\begin{aligned} A^2 + B^2 \rightarrow & 4[(\psi_5 + \psi_6 + \psi_7 + \psi_8 - 4\psi_0)^2/4 \\ & - \{(\psi_6 + \psi_8 - 2\psi_0)(\psi_5 + \psi_7 - 2\psi_0) \\ & - (\psi_1 + \psi_3 - \psi_2 - \psi_4)^2\}]/\{d^2(1 + \cos^2 \alpha)\}^2 \end{aligned}$$

To keep the mathematical consistency of the Eq. (5), the difference equation for C was also determined with the help of Fig. 3 and C takes the form as,

$$\begin{aligned} C \rightarrow & [(f_5 - f_0)(\psi_5 + \psi_6 - 2\psi_0) + (f_7 - f_0) \\ & (\psi_7 + \psi_8 - 2\psi_0)]/d^2(1 + \cos^2 \alpha) \end{aligned}$$

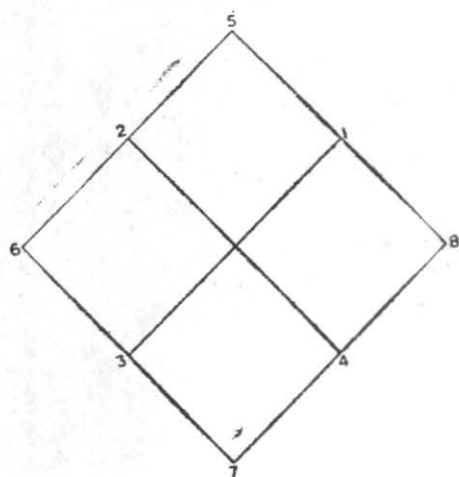


Fig. 3

4.3. Solution of difference equation

The technique of solving the difference equation is the same as suggested by Miyakoda (1960) which Shuman (1957) refers as 'Cycle Scan' method, with slight variation. Eq. (5) may be written as,

$$\nabla^2 \psi = -f + [2g \nabla^2 z + (A^2 + B^2) - C]^{\frac{1}{2}} \quad (8)$$

where g is the value of gravity which is taken as 9.8 m sec^{-2} . The R.H.S. of Eq. (8) is the forcing function for the solution of the Poisson equation,

$$\nabla^2 \psi = \sigma \quad (9)$$

σ is calculated from Eq. (8) using ψ^n where ψ^n is the n th iterative value, and then, Poisson Eq. (9) is solved by relaxation method. The solution corresponds to ψ^{n+1} , then ψ^{n+1} is inserted into Eq. (8). This procedure is repeated until $(\psi^{n+1} - \psi^n)$ converges to a certain small value, which is discussed in the next section.

The initial guess of ψ was taken as,

$$\psi = (g/\bar{f})z$$

where \bar{f} is the mean coriolis parameter of the region.

4.4. Grid length and convergence criterion

Different grid lengths were used in the two cases with latitude correction. In Case I, the length used was 3° latitude and in Case II the grid length used was 2.5° latitude.

The convergence criterion for (9) was taken to be

$$\frac{\bar{f}}{g} (\psi^{n+1} - \psi^n) < 10 \text{ cm}$$

4.5. Boundary conditions

The values of ψ at the boundaries were prescribed as $\psi = (g/f)z$ for solving the balance equation as depicted in Eq. (8).

5. Discussion

The balance- ψ was computed for two synoptic situations and the same was used to determine the balance wind components (u_b and v_b). The geostrophic wind components (u_g and v_g) were also computed from the original data. The following comparisons were made.

- (i) Zonal and meridional components of the actual wind against the geostrophic wind based on the original geopotential field.
- (ii) Zonal and meridional components of the actual wind against the computed balance wind.

The comparisons were made in terms of the root mean square deviations (i) along the latitudes, (ii) along the longitudes and (iii) for the whole area. These deviations have been summarised in Tables 1 and 2 for Cases I and II respectively. A brief discussion of the results is given below for each case.

5.1. Case I : 19 May 1971 at 00 GMT

5.1.1. Actual wind against geostrophic wind—

- (i) The Root Mean Square Deviations (R.M.S.D.) between the actual wind components and the geostrophic wind components are larger for the lower latitudes, *i. e.*, for latitudes 8°N and 11°N and (ii) the R.M.S.D. of the meridional components are larger in comparison to the deviations of the zonal components, both latitude and longitude-wise as well as for the whole area.

5.1.2. Actual wind against computed balance

- (i) The R.M.S.D. of the zonal components as well as the meridional components are almost equal along the latitudes, the longitudes as well as for the whole area and (ii) the R. M. S. D. between the actual wind and the balance wind are about 33 per cent less for the zonal components and about 50 per cent less for the meridional component in comparison to the R. M. S. D. between the corresponding components of the actual wind and the geostrophic wind.

5.2. Case II : 7 July 1963 at 12 GMT

5.2.1. Actual wind against geostrophic wind—

- (i) The R. M. S. D. of the zonal components are somewhat larger in comparison to the R.M.S.D. of the meridional components along the latitudes,

TABLE 1(a)
R.M.S.D. from actuals for Case I latitudinally

Component	Lat. ($^{\circ}$ N)											For whole region
	8	11	14	17	20	23	26	29	32	35	38	
<i>Geostrophic wind components</i>												
$(\overline{u_g - u})^2$	22.8	25.5	5.8	9.4	6.6	10.9	7.5	8.2	3.8	5.7	8.0	10.4
$(\overline{v_g - v})^2$	35.8	22.4	14.3	13.8	9.6	6.5	8.6	10.1	3.9	3.7	3.5	15.12
<i>Balance wind components</i>												
$(\overline{ub - u})^2$	7.2	3.3	3.5	7.5	9.6	11.5	6.3	8.3	7.5	6.2	6.2	7.4
$(\overline{vt - v})^2$	6.4	6.3	7.4	10.5	9.5	8.6	9.7	8.1	5.6	5.5	4.4	7.7

TABLE 1(b)
R.M.S.D. from actuals for Case I longitudinally

Component	Long. ($^{\circ}$ E)											for whole region
	66	69	72	75	78	81	84	87	90	93	96	
<i>Geostrophic wind components</i>												
$(\overline{u_g - u})^2$	14.0	6.4	7.0	7.9	10.0	11.1	7.9	7.4	6.5	15.8	14.7	10.4
$(\overline{v_g - v})^2$	16.2	19.6	16.9	16.6	12.7	7.8	8.8	12.5	21.7	18.3	10.4	15.2
<i>Balance wind components</i>												
$(\overline{ub - u})^2$	9.6	6.9	5.5	7.2	9.9	9.6	7.4	5.2	3.4	4.9	8.2	7.4
$(\overline{vt - v})^2$	9.8	9.8	9.8	9.9	9.8	6.0	8.6	5.0	5.1	6.4	4.6	7.7

the longitudes as well as for the whole area and (ii) the R. M. S. D. are of smaller magnitude in comparison to the corresponding R. M. S. D. in Case I.

5.2.2. *Actual wind against computed balance wind*—(i) The R. M. S. D. of the zonal and meridional components are of comparable magnitudes along the latitudes, the longitudes as well as for the whole area and (ii) the R. M. S. D. for the zonal components is lower by about 15 per cent and that of meridional component is lower by about 10 per cent in comparison to the corresponding R.M. S.D. for the geostrophic case.

In the study of Case I, for the actual wind against the geostrophic wind, the meridional components are found to be larger in comparison to the zonal components. This is so, because the flow is strongly meridional over the area under the influence of a large amplitude trough in the westerlies, whereas such a synoptic feature is absent in the Case II.

6. Conclusion

The important result of this study is that in both the cases we find the balance wind is a better approximation of the actual wind in comparison to the geostrophic wind as expected. This also shows that the numerical procedure adopted for solving the balance equation is quite satisfactory and has given the expected results even when we started from the geopotential field, which is known for its weak gradient and incorrect values in the tropics.

Thus, it is concluded that the non-divergent wind which is obtained by following this procedure approaches more nearly to the actual wind in comparison to the geostrophic wind. This wind, so determined, when used in the forecast models is expected to improve the numerical forecast in the Indian region.

Acknowledgements

We are very much thankful to Shri D. R. Sikka, Senior Scientific Officer of Indian Institute of

TABLE 2(a)
R.M.S.D. from actuals for Case II latitudinally

Component	Lat. ($^{\circ}$ N)												For whole region
	12.5	15.0	17.5	20.0	22.5	25.0	27.5	30.0	32.5	35.0	37.5	40.0	
$(u_g - u)^2$	<i>Geostrophic wind components</i>												
	7.4	7.3	10.4	6.3	7.7	8.3	9.4	6.6	12.7	10.4	9.2	13.5	9.3
$(v_g - v)^2$	6.5	4.7	3.4	3.5	5.1	6.2	4.8	11.0	11.3	7.6	8.0	9.0	7.2
	<i>Balance wind components</i>												
$(u_b - u)^2$	9.9	8.7	9.7	3.0	4.2	5.5	5.9	6.6	9.4	8.7	9.1	10.3	7.9
$(v_b - v)^2$	6.7	4.6	3.2	2.7	4.8	6.0	7.3	7.9	10.1	7.7	5.6	8.6	6.5

TABLE 2(b)
R.M.S.D. from actuals for Case II longitudinally

Component	Long. ($^{\circ}$ E)																For whole region	
	55.0	57.5	60.0	62.5	65.0	67.5	70.0	72.5	75.0	77.5	80.0	82.5	85.0	87.5	90.0	92.5		95.0
	<i>Geostrophic wind components</i>																	
$(u_g - u)^2$	3.8	5.2	5.5	6.2	8.0	10.0	12.1	10.0	9.7	9.7	10.4	10.2	9.6	10.1	10.5	11.5	11.0	9.3
$(v_g - v)^2$	4.0	3.7	4.4	4.0	8.0	9.9	9.5	7.8	8.7	9.2	9.8	8.2	6.2	4.5	5.8	5.4	8.2	7.2
	<i>Balance wind components</i>																	
$(u_b - u)^2$	4.0	3.9	2.7	2.7	4.4	6.5	8.9	9.0	8.5	9.0	9.2	9.4	9.0	9.7	9.5	10.2	10.3	7.9
$(v_b - v)^2$	5.0	2.3	2.2	4.0	8.2	9.1	10.2	7.7	7.7	7.0	7.7	6.4	6.1	6.1	6.1	5.4	6.5	6.5

Tropical Meteorology, who has helped in every phase of this paper. Our thanks are also due to Shri Y. Ramanathan, Meteorologist, NHAC, New Delhi for his valuable help in course of de-

bugging the program. The authors are greatly indebted to Dr. P. K. Das, Dy. Director General of Observatories (Planning) for his encouragement and keen interest in the problem.

REFERENCES

- Bolin, B. 1955 *Tellus*, 7, pp. 27-49.
 1956 *Ibid.*, 8, pp. 61-75.
 Bring, A. and Charasch, E. 1958 *Ibid.*, 10, pp. 88-94.
 Bushby, F. H. and Huckel, V. M. 1956 *Quart. J. R. met. Soc.*, 82, pp. 409-418.
 Charney, J. G. 1955 *Tellus*, 7, pp. 22-26.
 Miyakoda, K. 1956 *J. met. Soc. Japan*, 34, pp. 364-367.
 1960 Numerical solution of the balance equation, Tech. Rep. Japan Met. Agency, 3.
 Petterssen, S. 1956 *Weather Analysis and Forecasting*, 1, (Second Ed.).
 Ramanathan, Y., Sikka, D. R. and Dixit, C. M. 1971 *Indian J. Met. Geophys.*, 22, pp. 75-80.
 Shuman, F. G. 1957 *Mon. Weath. Rev.*, 85, pp. 329-332.