Indian J. Met. Hydrol. Geophys. (1975), **26**, 3, 320-322

550.342

Rossby wave and pure rotational wave

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ABSTRACT. In this note the proof of the relation that exists at the equator between speed of a pure rotational wave and wave lengths of streamlines and trajectory (Asnani 1972) has been provided. Also the proof of the idea of Constant Absolute Spin (C.A.S.) trajectory as distinct from Constant Absolute Vorticity (C.A.V.) trajectory developed is presented.

1. Introduction

In a short note entitled 'Rossby wave and Pure rotational wave' Asnani (1972) presented the result without giving the proof that at the equator, the speed C of a pure rotational wave is given by

$$U - C = \beta L_s L_T / 4\pi^2 \tag{1}$$

where L_s and L_T are the wave lengths of the streamline and the trajectory respectively. In the same note, the author also developed the idea of Constant Absolute Spin (C.A.S.) trajectory as distinct from Constant Absolute Vorticity (C.A.V.) trajectory. In the present paper a proof for these statements is provided.

2. Pure rotational wave

Let us consider pure inertial motion of a particle on a frictionless horizontal surface of the rotating earth in a β-plane approximation. Specifications of the initial condition are :

U, v_0 = Zonal and meridional components of velocity at the equator at $t=0$

α_0 = Angle which the particle makes with the equator at the time of crossing into the northern hemisphere.

$$V_0 = \sqrt{U^2 + v_0^2}$$

For simplicity, we shall consider cartesian system of co-ordinates rotating with the earth. x is positive towards east, y is positive towards north; $f = \beta y$

2.1. It is then easy to derive the following relationships (Whipple 1917; Rossby 1940; Wiin-Nielsen 1970) :

$$u = U + \tfrac{1}{2}\beta y^2 \tag{2}$$

$$v = [v_0^2 - U\beta y^2 - \tfrac{1}{4}\beta^2 y^4]^{\frac{1}{2}} = [V_0^2 - u^2]^{\frac{1}{2}} \tag{3}$$

$$\frac{du}{dt} = \beta y v = f v \tag{4}$$

$$\frac{dv}{dt} = -\beta y\left(U^2 + \tfrac{1}{2}\beta y^2\right) = -fu \tag{5}$$

$$A_T = 2\sqrt{V_0/\beta}\ \sin(\alpha_0/2) \tag{6}$$

$$L_T = 8\sqrt{\frac{V_0}{\beta}}\left[E\left(\sin\frac{\alpha_0}{2}, \frac{\pi}{2}\right) - \frac{1}{2}F\left(\sin\frac{\alpha_0}{2}, \frac{\pi}{2}\right)\right] \tag{7}$$

$$\Gamma = \frac{4}{\sqrt{V_0 \beta}} F\left(\sin\frac{\alpha_0}{2}, \frac{\pi}{2}\right) \tag{8}$$

A_T is the amplitude of the trajectory, *i.e.*, the maximum distance from the equator to which the particle projected at the equator travels northwards; L_T is the wave-length of the trajectory in x-direction. F and E are the elliptic integrals of the first and second kind respectively defined by

$$E(k, \theta) = \int_0^\theta \sqrt{(1 - k^2 \sin^2\theta)}\, d\theta \tag{9}$$

$$F(k, \theta) = \int_0^\theta \frac{d\theta}{\sqrt{(1 - k^2 \sin^2\theta)}} \tag{10}$$

Γ is the period in which the particle completes one oscillation at the equator.

2.2. In respect of the curvature of the path of ~~the particle~~, the following relationships are interesting.

$$\frac{dy}{dx} = \frac{v}{u} \tag{11}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{V_0^2}{u^2} \tag{12}$$

SECTION II

SHORT RANGE FORECAST MODEL FOR NUMERICAL WEATHER PREDICTION

$$\frac{d^2y}{dx^2} = -\frac{\beta y V_0^2}{u^3} \tag{13}$$

$$\frac{1}{R} = \frac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}} = -\frac{\beta y}{V_0} \tag{14}$$

$$\beta y + \frac{V_0}{R} = 0 \tag{15}$$

The curvature of the particle is proportional to its displacement from the equator. The path of the particle is the famous elastica, the shape of a thin elastic rod strained by forces applied at the two ends. As seen from Eq. (15), along the trajectory of the particle, the value of the absolute spin (coriolis term + curvature part of the spin) around the vertical is constant; the particle conserves its absolute spin. Hence its trajectory can be called C.A.S. (Constant Absolute Spin) trajectory. *A particle projected horizontally at the equator traces a C.A.S. trajectory.*

3. Trajectory with a small angle

So far, we have not placed any restrictions on the relative magnitudes of U and v_0 and hence on α_0. We now restrict α_0 to be a small angle. In other words $U >> v_0$. Now the trajectory becomes a sinusoidial curve, for which mathematical analysis becomes much simpler. It can be shown that under these circumstances, we have

$$A_T = \alpha_0 \sqrt{\frac{V_0}{\beta}} \doteq \alpha_0 \sqrt{\frac{U}{\beta}} \tag{16}$$

$$L_T = 2\pi \sqrt{\frac{V_0}{\beta}} \doteq 2\pi \sqrt{\frac{U}{\beta}} \tag{17}$$

$$\Gamma = \frac{2\pi}{\sqrt{V_0\beta}} \doteq \frac{2\pi}{\sqrt{U\beta}} \tag{18}$$

$$\frac{1}{R_T} = -\frac{4\pi^2}{L_T^2} y_T = -\frac{\beta}{V_0} y_T \doteq -\frac{\beta}{U} y_T \tag{19}$$

$$x_T \doteq Ut \tag{20a}$$

$$y_T = A_T \sin \frac{2\pi}{L_T} x_T \tag{20b}$$

$$v_0 = 2\pi U \frac{A_T}{L_T} \tag{21}$$

$$v_T = v_0 \cos\left(\frac{2\pi}{L_T} x_T\right) \tag{22}$$

$$\frac{d}{dt}\left(\frac{V_0}{R_T} + f\right) \doteq 0 \tag{23}$$

4. Set of particles

We now visualise a set of suitably synchronised particles projected from the equator, all with the same velocity of projection V_0 and making the same small angle α_0 with the equator, but crossing the equator with a time lag, one behind the other. These trajectories will give rise to a streamline pattern in the form of a wave with the following characteristics (Rossby 1940, Haltiner and Martin 1957, p. 186).

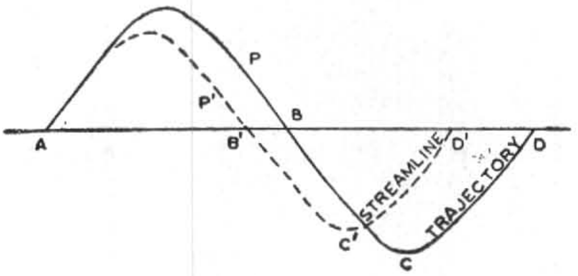

Fig. 1. Trajectory and streamlines

$$\frac{A_s}{A_T} = \frac{U-C}{U} \tag{24}$$

$$\frac{L_s}{L_T} = \frac{U-C}{U} \tag{25}$$

$$\psi = -Uy + \frac{v_0 L_s}{2\pi} \sin \frac{2\pi}{L_s}\left(x-Ct\right) \tag{26a}$$

The central streamline $\psi=0$ being given by

$$y_s = \frac{v_0}{U} \frac{L_s}{2\pi} \sin \frac{2\pi}{L_s}\left(x-Ct\right) \tag{26b}$$

$$A_s = \frac{v_0}{U} \frac{L_s}{2\pi} \tag{27}$$

In Fig. 1, the trajectory and the streamline are shown schematically.

APBD is the trajectory and AP′ B′ D′ is the streamline. During one complete period, a particle moves along the trajectory from A to D *via* P, B, C. During the same period, the streamline wave moves through the distance D′ to D (distance$=L_T-L_s$). Corresponding to the position P of the moving particle on its trajectory, there is a point P′ on the moving streamline such that P and P′ are always on the same ordinate and further the direction of movement of P is the same as the direction of the streamline at P′, *i.e.*, slopes of the trajectory and streamline are identical at 'corresponding' points. It can also be shown that the speeds at 'corresponding' points are identical but curvatures are different. Hence the spin of the particle V_T/R_T is different from the vorticity V_T/R_s of the streamline.

5. Speed of the pure rotational wave

From Eqs. (17) and (18), we get

$$\Gamma = 4\pi^2/\beta L_T \tag{28}$$

From (20a), the period Γ is

$$\Gamma = L_T/U \tag{29}$$

From (28) and (29), we get

$$U = \beta L_T^2/4\pi^2 \tag{30}$$

Combining with Eq. (25), we get

$$U - C = \beta L_s L_T/4\pi^2 \tag{31}$$

It will be seen that this wave formula is different from Rossby wave formula given below.

$$U - C = \frac{\beta L_s^2}{4\pi^2} \tag{32}$$

Eqs. (31) and (32) become identical only for a stationary wave ($C=0$; $L_s = L_T$).

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