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Numerical solutions for a two dimensional mountain wave model

U. S. DE

Meteorological Office, Poona

and

S. M. T. FAROOQUI and (Miss) A. G. DEODHAR

Indian Institue of Tropical Metcorology, Poona

ABSTRACT. Numerical solutions for a two-dimensional linear mountain wave model for a baroclinic compressible atmosphere have been obtained. The techniques developed for smaller obstacles have been suitably adopted for application to extended topographic features with real air stream characteristics. Computations for different situations when mountain wave observations were available have been made. Important aspects of the results have been discussed.

1. Introduction

The mountain wave problem in two and three dimensions have till recently been tackled analytically. It is, therefore, quite restricted in application to real atmospheric conditions. Sawyer (1960) was, perhaps, the first to attempt the integration of Scorer's equation by a quasi-numerical technique. Later, Krishnamurty (1964) used a two-dimensional model in x-s plane (where, s is entropy) to obtain the solutions by a numerical marching scheme. Pekelis (1966, 1969) and Onishi (1969) have also attempted to obtain the solutions by numerical methods. However, these methods except for the one by Sawyer has not been used for extended topographic obstacles under real atmospheric conditions. Sarker (1967) and De (1973) have used the above quasi-numerical method for real conditions for the Western Ghats and the Assam hills respectively. We present in this paper a fully numerical technique of solving the two-dimensional mountain wave equation. The method is similar to that by Pekelis (1969).

2. Governing equations

We consider a two-dimensional (x-z) steady state linear model in a compressible inviscid atmosphere. If a westerly airstream with velocity U (z), temperature T (z), density ρ (z) and pressure P (z) is subjected to a small perturbation such that the perturbation quantities are represented by u (x, z), T (x, z), ρ (x, z) and p (x, z), we can linearise the equations. The equation for the perturbation vertical velocity w (x, z) for such a system can be written as,

$$\left(1-\frac{U^2}{C^2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}-\left(\frac{g}{RT}-\frac{\chi R\gamma-2U dU/dz}{C^2-U^2}\right)\frac{\partial w}{\partial z}+\left[\frac{(\chi-1)g^2}{C^2U^2}-\frac{g\chi R\gamma}{(C^2-U^2)U^2}-\frac{1}{U}\frac{d^2U}{dz^2}+\left(\frac{\chi g}{C^2-U^2}-\frac{\chi R\gamma+2g}{C^2U^2}\right)\frac{1}{U}\frac{dU}{dz}-\frac{2}{C^2-U^2}\left(\frac{dU}{dz}\right)^2\right]w=0 \qquad (1)$$

Here $C^2=\chi RT$, $\chi=c_p/c_v$, ratio of specific heats at constant pressure and at constant volume. R is gas constant C is velocity of sound, g is acceleration due to gravity. $\gamma=-(dT/dz)$ is the lapse rate. We introduce the following for brevity

$$M(z)=1-\frac{U^2}{C^2}\ ,\ M'(z)=-\frac{2U}{C^2}\ \frac{dU}{dz}$$

$$\sigma(z)=\frac{\gamma^*-\gamma}{T},$$

γ^* is the dry ad abatic lapse rate.

$$S(z)=\frac{1}{\rho}\ \frac{d\rho}{dz}=\frac{g-R\gamma}{RT} \qquad (2)$$

$$o(z)=S(z)-g/C^2$$

Assuming $M(z) \simeq 1$ for wind speeds of the order of $U = 50$ m sec^{-1} we get,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} - \left\{ M'(z) + S(z) \right\} \frac{\partial w}{\partial z}$$

$$+ \left[\left\{ M'(z) + \sigma(z) \right\} \frac{g}{U^2} - \frac{1}{U} \frac{d^2 U}{dz^2} \right.$$

$$\left. + \left\{ M'(z) + S(z) \right\} \frac{1}{U} \frac{dU}{dz} \right] w = 0 \qquad (3)$$

The equation of continuity can be written as,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{g}{C^2} w = 0 \qquad (4)$$

Accordingly we introduce stream function ψ defined as:

$$\frac{\partial \psi}{\partial x} = - w \exp \left[- \int_0^z \frac{g}{C^2} d\xi \right]$$

$$\frac{\partial \psi}{\partial z} = (U + u) \exp \left[- \int_0^z \frac{g}{C^2} d\xi \right] \qquad (5)$$

The distribution of $\psi(x, z)$ in basic flow is given by

$$\overline{\Psi}(z) = \int_0^z U(\xi) \exp \left[- \int_0^\xi \frac{g}{C^2(\eta)} d\eta \right] d\xi \qquad (6)$$

Equation (3) then transforms into,

$$\frac{\partial^2 \psi}{\partial x'} + \frac{\partial^2 \psi}{\partial z^2} - \left(M' + \sigma - \frac{g}{C^2} \right) \frac{\partial \psi}{\partial z}$$

$$+ \left\{ (M' + S) \frac{1}{U} \frac{dU}{dz} (M' + \sigma) \frac{g}{U^2} \right.$$

$$\left. - \frac{gT'}{C^2 T} - \frac{1}{U} \frac{d^2 U}{dz^2} \right\} \psi = \frac{\partial^2 \overline{\Psi}}{\partial z^2}$$

$$- \left(M' + \overline{\sigma} - \frac{g}{C^2} \right) \frac{\partial \overline{\Psi}}{\partial z} + \left[(M' + S) \frac{1}{U} \frac{dU}{dz} \right.$$

$$\left. + (M' + \overline{\sigma}) \frac{g}{U'} - \frac{gT'}{C^2 T} - \frac{1}{U} \frac{d^2 U}{dz^2} \right] \psi \qquad (7)$$

We introduce dimensionless variables defined by

$$\overline{x} = \frac{x}{H}, \ \overline{z} = \frac{z}{H}, \ \overline{\Psi} = \frac{\psi}{U(0) H}$$

where, H is the length scale selected.

The equation then reduces to,

$$\frac{\partial \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} - K(z) \frac{\partial \psi}{\partial z} + F(z) \psi$$

$$= \frac{\partial^2 \overline{\Psi}}{\partial z^2} - K(z) \frac{\partial \overline{\Psi}}{\partial z} + F(z) \overline{\Psi} \qquad (8)$$

where,

$$F(z) = (M' + SH) \frac{1}{U} \frac{dU}{dz} + (M' + \sigma H) \frac{gH}{U^2}$$

$$- \frac{gH}{C^2} \frac{T'}{T} - \frac{1}{U} \frac{d^2 U}{dz^2},$$

$$K(z) = M' + \sigma H - \frac{gH}{C^2} \qquad (9)$$

For the sake of convenience we drop the wave sign in further part of the paper, although the variables will be the dimensionless quantities as defined above. Equation (8) is the final mountain wave equation for our model in terms of the dependent variable $\psi(x, z)$.

Boundary conditions

Equation (8) is to be solved for a finite region for which following boundary conditions are used.

(*i*) *The upstream boundary condition*: We assume that far in advance of the obstacle the motion approaches the undisturbed flow.

$$\psi(x, z) \rightarrow \overline{\Psi}(z) \quad \text{when } x \rightarrow - \infty \qquad (10)$$

and in the region of integration the motion is finite, *i.e.*,

$$|\psi(x, z)| < \infty \qquad - \infty < x < \infty$$

(*ii*) *The lower boundary condition*: We assume that the flow at ground is tangential to the surface. On linearisation, we get

$$w(x, 0) = U(0) \frac{\partial}{\partial x} \xi(x)$$

where, $\xi(x)$ is the ground profile.

This gives, $\left. \dfrac{\partial \psi}{\partial x} \right|_{z=0} = - \dfrac{U(0)}{U(0)} \cdot \dfrac{\partial \xi(x)}{\partial x}$

or $\quad \psi(x, 0) = - \xi(x) \qquad (11)$

Thus the ground profile is a streamline.

(*iii*) *Upper boundary condition*: In the present work we assume that at height L there is

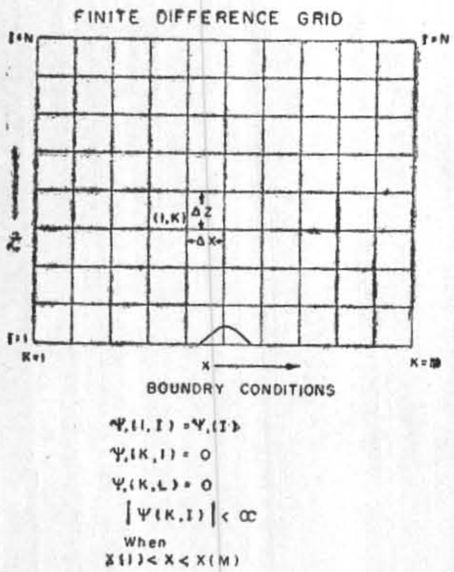

Fig. 1. Area of integration for the problem

no perturbation, *i.e.*,

$$\psi(x, L) = \overline{\Psi}(L) \tag{12}$$

and the height L being non-dimensional with respect to H. It may be seen that equations (10) to (12) are non homogeneous. These condition. can be made homogeneous by the following transformation—

$$\psi_1(x,z) = \psi(x,z) + \left(1 - \frac{z}{L}\right)\xi(x) - \frac{z}{L}\,\overline{\Psi}(L) \tag{13}$$

which is one of the requirements for solving boundary value problem as eigen value problem. Then equation (8) is rewritten in terms of the new variable as

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial z^2} - K(z)\,\frac{\partial\psi_1}{\partial x} + F(z)\,\psi_1 = G(x,z) \tag{14}$$

where,

$$G(x,z) = \frac{\partial^2\psi_1}{\partial z^2} - K(z)\,\frac{\partial\psi_1}{\partial z_2} + F(z)\,\psi_1 + \left(1 - \frac{z}{L}\right)\left\{\frac{\partial^2\xi}{\partial x^2} + F(z)\,\xi(x)\right\} + \frac{K(z)}{L}\,\xi(x)$$

and the boundary conditions (10) to (12) become,

$$|\psi_1(x,z)| < \infty \qquad \text{when } -\infty < x < \infty$$

$$\psi_1(x,z) \to \overline{\Psi}_1(z) \qquad \text{when } x \to -\infty \tag{15}$$

and $\psi_1(x, 0) = \psi_1(x, L) = 0$

In (14) and (15) $\overline{\Psi}_1(z)$ is given by

$$\overline{\Psi}_1(z) = \overline{\Psi}(z) - \frac{Z}{L}\,\overline{\Psi}(L) \tag{16}$$

3. Numerical solution

The numerical integration is carried over a region extending on either side of the obstacle. The area is shown in Fig. 1. The vertical grid length along z-axis is $\triangle z = L/N$ where N is number of levels and L is the height of the upper boundary.

We replace the derivatives with respect to z in Eq. (14) by finite differences and re-write Eq. (14) as

$$\psi_1''(1) + A_1\,\psi_1(1) + B_1\,\psi_1(2) = G_1$$

$$\psi_1''(\ ) + C_2\,\psi_1(1) + A_2\,\psi_1(2) + B_2\,\psi_1(2) = G$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

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$$\psi_1''(N\text{-}1) + C_{N-1}\,\psi_1(N\text{-}2) + A_{N-1}\,\psi_1(N-1) = G_{N-1} \tag{17}$$

General form of equation (17) is,

$$\psi_1''(i) + C_i\,\psi_1(i-1) + A_i\,\psi_1(i) + B_i\psi_1(i+1) = G_i,$$

$$i = 1, 2, \ldots\ldots N\text{-}1$$

$$\text{where, } \psi_1''(i) \equiv \left|\frac{\partial^2\psi_1}{\partial x^2}\right|_{z=z_i},\ i = 1, 2, \ldots\ldots N\text{-}1$$

$$A_i = F_i - \frac{2}{\triangle z^2}, \qquad F_i = F(z_i)$$

$$B_i = \frac{1}{\triangle z^2} - \frac{K_i}{2\triangle z}, \quad K_i = K(z_i)$$

$$C_i = \frac{1}{\triangle z^2} + \frac{K_i}{2\triangle z}$$

The system of equations (17) represent ordinary differential equations for N—1 levels. These may be reduced to a canonical form such that it becomes a system of N—1, independent equations in a variable $\phi_n(x)$ which are linear combinations of $\psi_1(n)$, $n=1, 2, \ldots, N-1$ provided the matrix,

$$[A] = \begin{bmatrix} A_1 & C_2 & 0 & & & & \\ B_1 & A_2 & C_3 & & & & \\ 0 & B_2 & A_3 & C_4 & & & \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & & & \\ & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \\ & & & & B_{N-3} & A_{N-2} & C_{N-1} \\ & & & & & 0\ B_{N-2} & A_{N-1} \end{bmatrix}$$

has real and distinct roots. In a system where B_i, and C_i are positive the eigen values are real and different. Consequently, the system (17) can be reduced to canonical form.

Let $[\alpha_{ij}]$ be an eigen vector/matrix and $\alpha_i = \{\alpha_{ij}\}$, $j = 1, 2, \ldots, N-1$ be an eigen vector corresponding to an eigen value λ_i. We denote,

$$\overline{\Psi_1} = \overline{\Psi}(x) = \{\psi_1(1), \psi_1(2), \ldots\ldots, \psi_1(N-1)\}$$

$$\overline{G} = \overline{G}(x) = \{G_1(x), G_2(x), \ldots\ldots, G_{N-1}(x)\}$$

$$\overline{\phi} = [\alpha_{ij}]\,\overline{\Psi_1}$$

Thereby Eq. (17) reduces to

$$\phi_i'' + \lambda_i \phi_i = \overline{\alpha_i}\,\overline{G_i} \tag{18}$$

The boundary conditions formulated in terms of the new variable ϕ are :

The undisturbed value of ϕ_i is

$$\phi_{i0} = \alpha_{i1}\overline{\Psi}(1) + \alpha_{i2}\overline{\Psi}_1(2) + \ldots\ldots + \alpha_{iN-1}\overline{\Psi}_1(N-1) \tag{19}$$

Here $\overline{\Psi_1}(k)$ are the values of $\psi_1(x,z)$ at the k^{th} level in the basic flow.

$$|\phi_i(x)| < \infty \qquad \text{where } -\infty < x < \infty$$

$$\phi_i(x) \rightarrow \phi_{i0} \qquad \text{when } x \rightarrow -\infty$$

Equation (18) is solved numerically to determine the values of $\phi(x)$.

4. Method of computation

The computations were carried out with the help of a high speed electronic computer. To start with, from the data of wind and temperature in the basic flow, the parameters $K(z)$, $F(z)$ and stream functions $\overline{\Psi}(z)$ are calculated by using equations (6) and (9) respectively. The matrix $[A]$ is also set up by calculating the values of its elements from the above parameters. The eigen values of $[A]$ are obtained by using a standard programme based on 'Strum sequence' property. The corresponding eigen vectors are determined by means of iteration. All the eigen values of $[A]$ do not represent the real waves in the atmosphere of our interest. For this purpose we neglect the eigen values which correspond to wavelengths > 50 km. The choice depends upon the nature of investigation, and the scale factor H. From the eigen values ϕ_{i0} is computed using equation (19). At the levels for which the eigen values have been neglected ϕ_{ik} is put equal to ϕ_{i0}. However, for computation of ϕ_{ik} at other levels we proceed as follows.

For $\lambda_i \geqslant 0$ we assume that $\phi_{i1} = \phi_{i0}$ and the marchiug scheme adopted is

$$\phi_{ik+1} = (\alpha_i\,\overline{G})_k \triangle x^2 + (2 - \triangle x^2 \lambda_i)\phi_{ik} - \phi_{ik-1}$$

The marching scheme is stable for $\triangle x < 2/\sqrt{\lambda_i}$

For $\lambda_i \leqslant 0$ we write equation (18) in finite difference form

$$\phi_{ik+1} + \phi_{ik-1} - 2 + \phi_{ik} + \lambda_i \triangle x^2 \phi_{ik} - (\alpha_i G)_k \triangle x^2 = 0 \tag{21}$$

and is solved by iteration.

In equations (20) and (21) ϕ_{ik} represents the value of ϕ at ith vertical level and kth horizontal level. Thus ϕ is obtained at each vertical level and for each horizontal grid point. Then, using a standard programme we invert the eigen vector matrix $[\alpha_{if}]$ to obtain the values of $\psi_1(x,z_i)$ from the computed ϕ field,

$$\psi_1(x, z) = [\alpha_{ij}]^{-1}\,\phi$$

Finally the stream function $\psi(x,z)$ is recovered from the transformation,

$$\psi(x, z) = \psi_1(x, z) - \left(1 - \frac{z}{L}\right)\xi(x) + \frac{z}{L}\,\overline{\Psi}(L)$$

at all grid points.

5. Results and conclusions

Using the method described above we have computed the streamline displacements for a few cases. The distribution of $F(z)$, $K(z)$ parameters and the associated streamline displacements are shown in Figs. 2 to 6. The computations have been made for a small bell-shaped obstacle (a=2 km, b= 1 km); a large bell-shaped obstacle (a=20 km, b=1 km) and the Assam hills. For large bell-shaped obstacle streamline displacement pattern are not shown here.

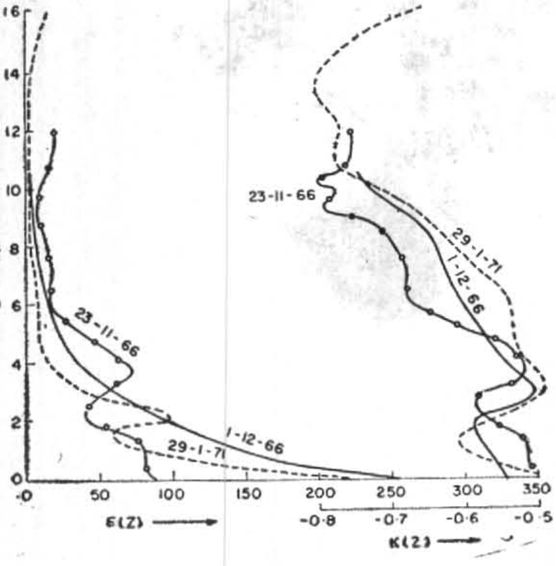
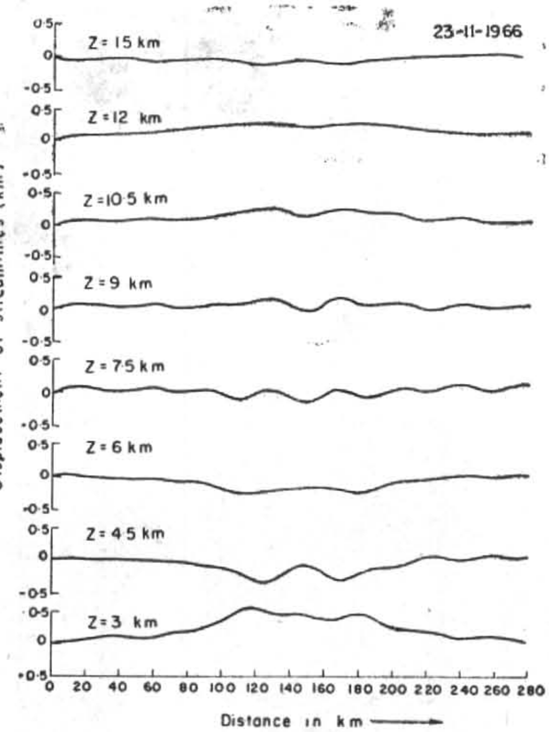


Fig. 2. $K(z)$ and $F(z)$ distribution



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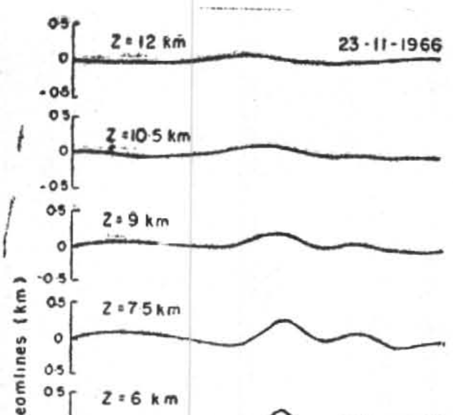

Fig. 3. Streamline displacement pattern for a small ridge (for bell shaped ridge $a=2$ km and $b=1$ km)

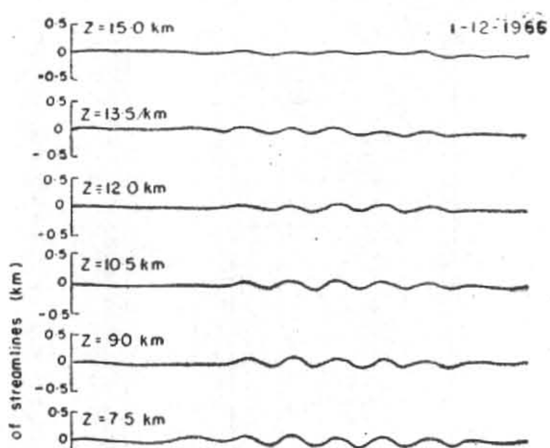

Fig. 5

Fig. 4 & 5. Streamline displacement pattern for Assam hills

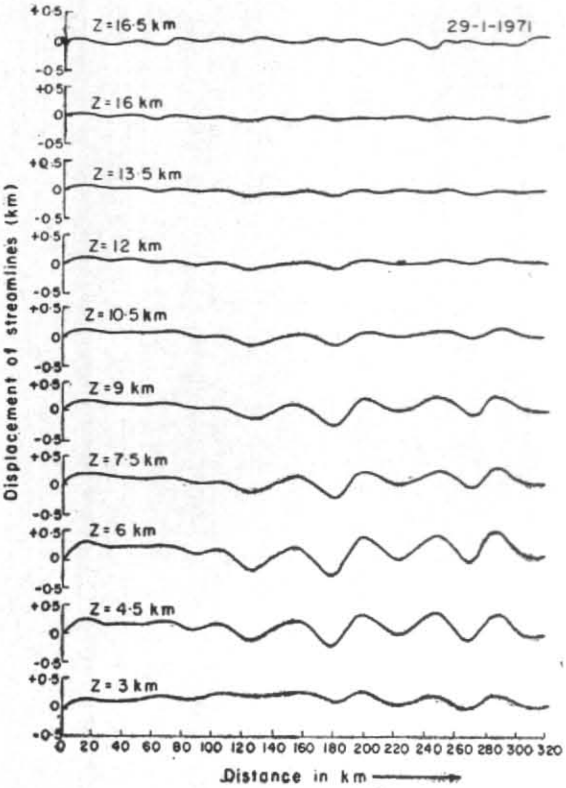

Fig. 6. Streamlines displacement pattern for Assam hills

The average elevation of Assam hills along west-east direction is represented by combining two ridges (Naga and Lushai hills) as

$$\xi(x) = \frac{a^2 b_1}{a^2+x^2} + \frac{a^2 b_2}{a^2+(x-d)^2}$$

where b_1 and b_2 are heights of two ridges, a is the half-width and d is the distance separating the two ridges. It is seen that the streamline displacements are confined to the downstream side only. On the upstream side in case of large obstacles there are some feeble oscillatory features which are not important. The vertical variation of the displacements is cellular in nature being alternatively positive and negative upto a height of about 10 to 15 km. The wavelengths associated with smaller obstacles are between 7 and 11 km and those associated with large obstacles are of the order of 30 km or more. For Assam hills the wavelengths computed by this method on 23 November 1966, 1 December 1966 and 29 January 1971, are aporoximately 27, 30 and 35 km respectively whereas those obtained by analytical and quasi-numerical methods (De 1973), for the first two cases were of order of 22 and 28 km. The absolute magnitude of the streamline displacement which are of the order of 200 m appear realistic. Satellite cloud pictures from ESSA-3, ESSA-8 and ITOS-1 support the existence of the wave cloud in Assam-Burma and neighbouring region during all these occasions. The observed wavelengths, as revealed by satellite pictures (De 1971) are 23 and 20 km on 23 November and 1 December 1966 respectively. For 29 January 1971 it is 30 km.

The following conclusions may be drawn from the study :

(*i*) The model appears suitable for studying mountain waves associated with real atmospheric conditions and actual ground relief of extended dimensions.

(*ii*) The results obtained by this method for small as well as large obstacles are in fair agreement with solutions obtained by analytical and quasi-numerical methods.

(*iii*) The results show that smaller obstacles give rise to streamline patterns with predominantly shorter wave lengths. On the other hand, the streamline patterns associated with large obstalcss show oscillations with predominantly larger wavelengths.

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REFERENCES

De, U. S.	1971	*Indian J. Met. Geophys.*, **22**, 3, pp. 361-364.
	1973	Sci. Rep. No. RR 007, Indian Inst. Trop. Met., Poona.
Krishnamurti, T. N.	1964	*Mon. Weath. Rev.*, **92**, 4, pp. 147-160.
Onishi, G.	1969	*J. met. Soc. Japan*, **47**, 5, pp. 352-359.
Pekelis, E. M.	1966	*Bull. (Izv). Acad. Sci.*, USSR, *atmos. & Ocean Phys.*, **2**, 11, pp. 1113-1125.
	1969	*Ibid.*, **5**, 1, pp. 1-7.
Sarker, R. P.	1967	*Mon. Weath. Rev.*, **95**, 10, pp. 673-684.
Sawyer, J. S.	1960	*Quart. J. R. met. Soc.*, **86**, pp. 326-345.