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A two dimensional mathematical model of a tropical cyclone

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ABSTRACT. A two dimensional mathematical model of a tropical cyclone is developed. It appears to represent closely the observed features of velocity, vorticity and divergence fields at surface in a tropical cyclone.

1. Introduction

The mathematical models based on Rankine's vortex and modifications thereof do not explain all the observed features of wind field in a tropical cyclone. Further, the theoretical soundness is questioned by Riehl (1954) whose book on *Tropical Meteorology* is referred to for an excellent discussion on this topic.

The author hypothesises that the tropical cyclonic storm constitutes a Helmholtz dynamical system. The characteristics of Helmholtz dynamical system have been established by the author in a separate un-published paper. On the basis of the hypothesis, a two dimensional mathematical model is proposed. With the help of a worked example, it is shown that the model fits well with the observed features of wind field in a tropical cyclone.

2. Velocity field at the suface in a tropical cyclonic storm

Ring of strong wind—The most striking feature of a tropical cyclonic storm is the ring of very strong wind around the centre. Fig. 1 is a schematic reproduction of total wind speed as given by Riehl (1950). The isotachs of speed have a horse-shoe pattern enclosing the region of maximum wind. The Maximum wind is different in different directions from the centre.

Strean line pattern—Fig. 2 is a schematic reproduction of streamlines in tropical cyclonic storm (based on Fig. 11.26 at p. 311 of Riehl's *Tropical Meteorology*). The two important features are: (*i*) in-flow, characteristics towards the centre of the storm and (*ii*) a 'col' region to the southwest of the centre.

Divergence field—Fig. 3 gives the profile of divergence (based on Fig. 11.7 at p. 290 of Riehl). Dotted line represents extrapolation by the author. The important features are (*i*) maximum of convergence at the centre, (*ii*) a ring of non-divergence around centre, (*iii*) maximum of divergence and (*iv*) the divergence vanishing towards the periphery of the storm.

Vorticity field—Fig. 4 is a schematic representation of vorticity and the dotted line is extrapolation and modification by the author. The solid line is based on Fig. 11.7 at pp. 290 of Riehl (1950). The important features are —

(*i*) Maximum vorticity at the centre,

(*ii*) Vorticity vanishing towards the periphery,

(*iii*) Minimum vorticity and

(*iv*) A ring of zero vorticity around the centre.

The author deviates from the observations given by Riehl in this aspect for reasons to be explained later.

For purposes of modelling, the important features as in Table 1 are taken. It may be noted that except in the vorticity field, all the features are based on observations.

3. Hypothesis

To define a dynamical system, either the force field or the potential field is required. It is hypothesised that the potential energy density V_m associated with the mean desnsity ρ_m of the fluid in the case of a two dimensional tropical cyclone is defined as :

$$V_m = (A^2 + B^2) \rho_m F \phi^2 \tag{1}$$

where A and B are constants,

$$\phi = \exp \left[- \left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \right]$$

$$F = - \left[\frac{x^2}{\sigma_1^4} + \frac{y^2}{\sigma_2^4} - \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right]$$

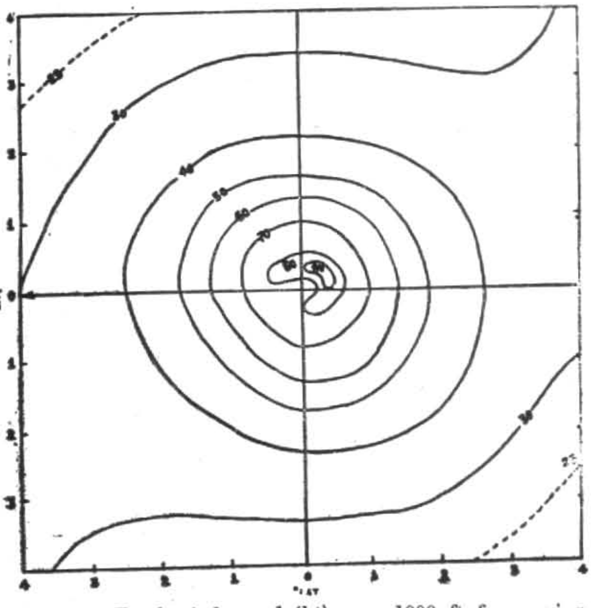

Fig. 1. Total wind speed (kt) near 1000 ft for moving storm. Arrows indicates movement of storm

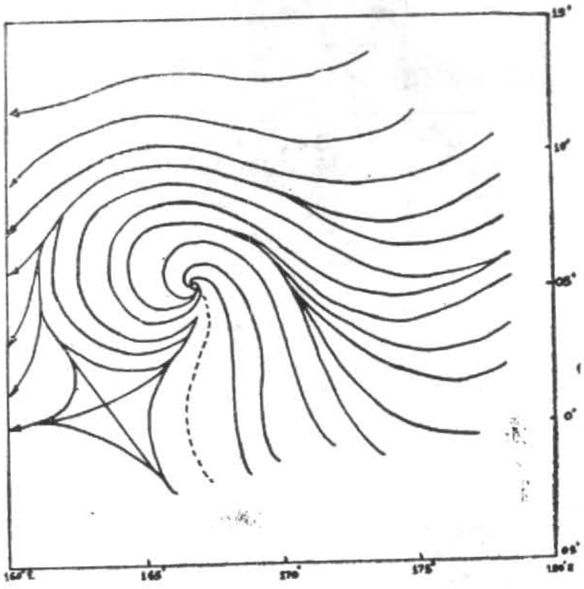

Fig. 2. Streamlines in a tropical cyclonic storm

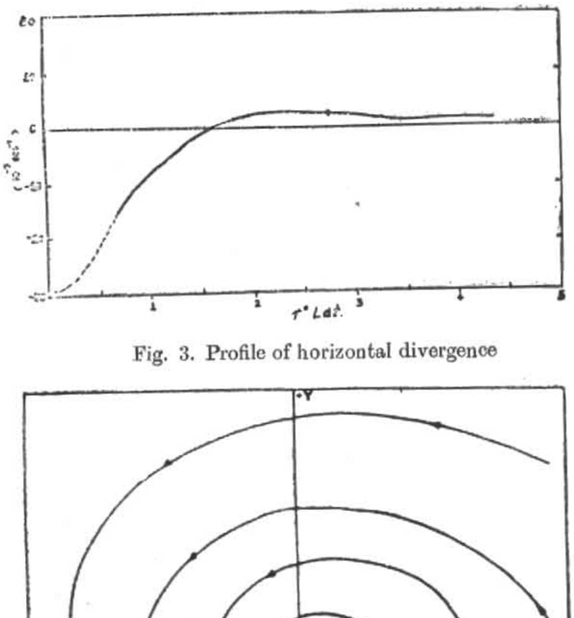

Fig. 3. Profile of horizontal divergence

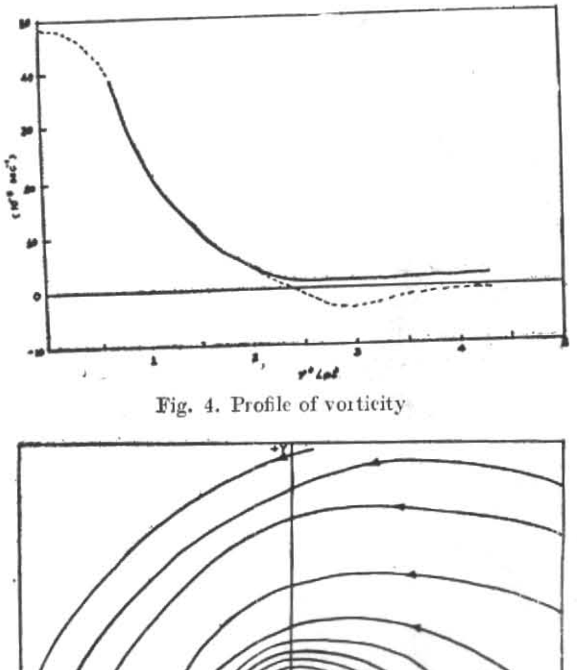

Fig. 4. Profile of vorticity

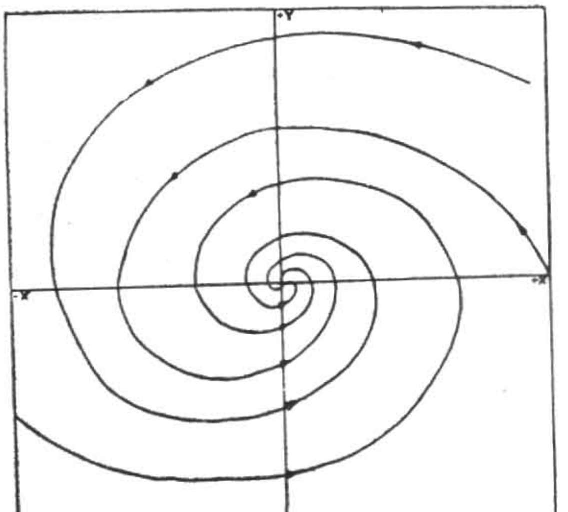

Fig. 5. Spiralling streamline pattern

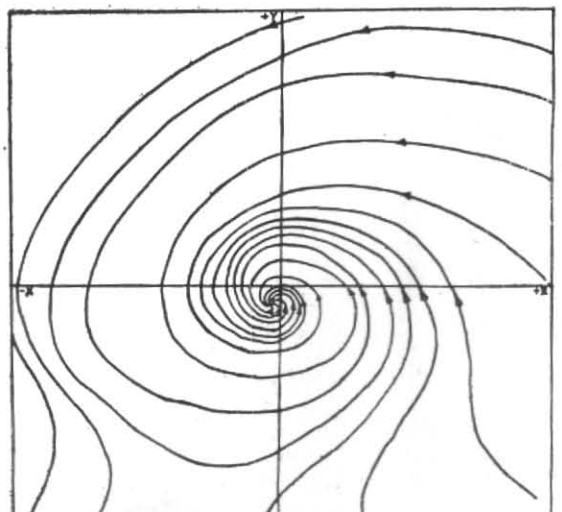

Fig. 6. Spiralling streamline pattern in a cyclone with superposed uniform E'ly current of 10 mps

TABLE 1

Character	Features	Remarks
Streamline pattern	(*i*) Spiralling flow inwards terminating at the centre	Based on observations
	(*ii*) A 'col' region to the southwest of centre	Do.
Isotach field	(*i*) A ring of strong winds	Do.
	(*ii*) Horse-shoe shaped isotachs enclosing maximum wind	Do.
Divergence	(*i*) Maximum convergence at the centre	Extrapolation by the author
	(*ii*) Vanishing towards the periphery	Based on observations
	(*iii*) Maximum of divegence	Do.
	(*iv*) A ring of non-divergence	Do.
Vorticity	(*i*) Maximum vorticity at the centre	Extrapolation by the author
	(*ii*) Vanishing towards the periphery	Based on observations
	(*iii*) A ring of zero vorticity arcund centre	Not observed. Author's deviation
	(*iv*) Minimum of vorticity with negative vorticity value	Do.

and σ_1, σ_2 are parameters characterising the potential energy density field. They can be evaluated from the ring of maximum wind. With this hypothesis, it is possible to show that the velocity field must be of the form

$$\mathbf{V} = A \nabla \phi + B (\nabla \phi \times \mathbf{k}) \quad (2)$$

where $\nabla = \mathbf{i} \dfrac{\partial}{\partial x} + \mathbf{j} \dfrac{\partial}{\partial y}$

i, **j**, **k** are unit constant vectors along x, y, z axes respectively.

The hypothesis can be accepted as tenable if the velocity field predicted on the basis of author's theory corresponds with the actual velocity field observed in a tropical storm.

4. Characteristics of the mathematical model

The components of velocity u and v are obtained from Eq. (2) and they are as in Eq. (3)

$$\left.\begin{aligned} u &= \left[-\frac{Ax}{\sigma_1^2} - \frac{By}{\sigma_2^2} \right] e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)} \\ v &= \left[\frac{Bx}{\sigma_1^2} - \frac{Ay}{\sigma_2^2} \right] e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)} \end{aligned}\right\} \quad (3)$$

specific values of A, B, σ_1 and σ_2 are chosen as under with a view to illustrate the characteristic of the model.

$$A = 2 \times 10^{10} \qquad B = 10 \times 10^{10}$$
$$\sigma_1 = \sqrt{2} \times 10^7 \qquad \sigma_2 = 1 \times 10^7$$

All units are in CGS system.

Using the above values of A, B, σ_1 and σ_2 in Eq. (3), u and v can be re-written as

$$\left.\begin{aligned} u &= (-x-10y)\, 10^{-4} \exp\left[-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \right] \\ v &= (5x-2y)\, 10^{-4} \exp\left[-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \right] \end{aligned}\right\} \quad (4)$$

u and v are in cm/sec.

Let a uniform easterly current u_m with a speed of 10^3 cm/sec be superposed on the cyclone. The velocity components (u^*, v^*) in this case will be

$$u^* = u + u_m \qquad v^* = v \quad (5)$$

Streamline pattern—Streamlines can be obtained by solving differential equation (7)

$$\frac{dy}{dx} = \frac{v}{u} = \frac{5x-2y}{-x-10y} \quad (7)$$

The pattern obtained is given in Fig 5.

It may be noted that the streamlines spiral inwards with a singular point at the centre of the co-ordinate system. When a uniform current is superposed, the streamline pattern can be obtained by solving Eq. (7)

$$\frac{dy}{dx} = \frac{v^*}{u^*} = \frac{v}{u+u_m} \quad (8)$$

Fig. 6 gives the pattern obtained by superposing a uniform easterly current of 10 m/sec. The effect of superposition is to shift the centre of the cyclone to the southwest of the original position. Further a 'col' region with a point of neutrality is seen in the same quadrant.

Isotach field—Squaring u and v components and adding, we get the square of speed $|V|^2$ as a function of x and y as in Eq (9).

$$|V|^2 = \left[\left(-\frac{Ax}{\sigma_1^2} - \frac{By}{\sigma_2^2} \right)^2 + \left(\frac{Bx}{\sigma_1^2} - \frac{Ay}{\sigma_2^2} \right)^2 \right] e^{-\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}\right)} \quad (9)$$

Expressing $|V|^2$ in polar co-ordinates, we get

$$|V|^2 = F(\theta)\, r^2\, e^{-r^2 f(\theta)} \quad (10)$$

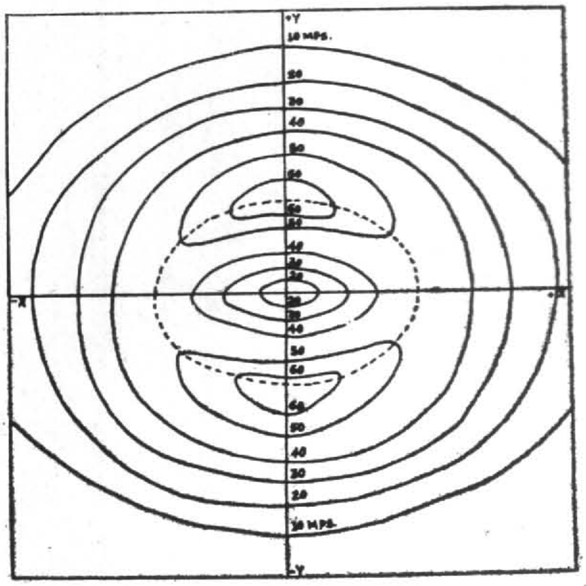

Fig. 7. Speed field in a cyclone. Dotted lines is ring of max. wind and continons lines are isotachs (m/sec)

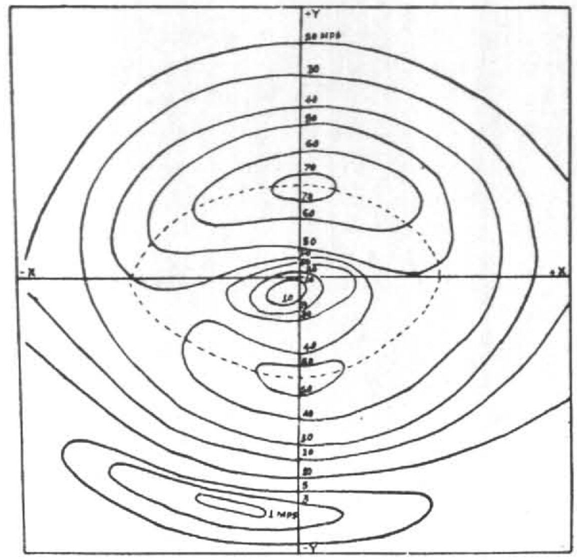

Fig. 8. Speed field in a cyclone with superposed uniform E'ly current of 10 mps

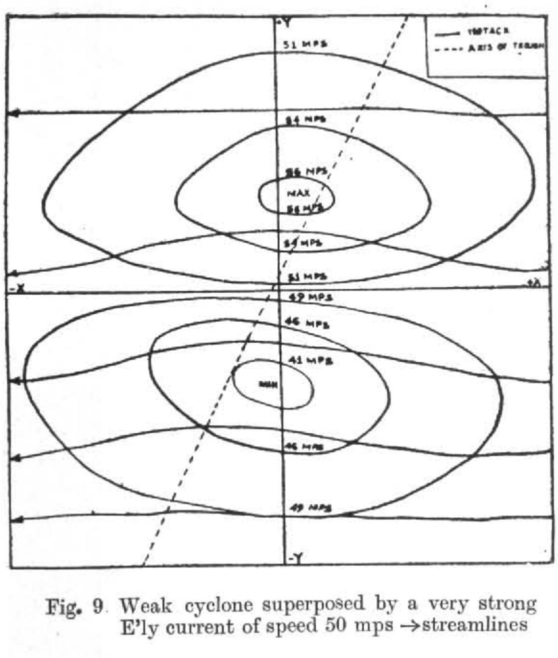

Fig. 9 Weak cyclone superposed by a very strong E'ly current of speed 50 mps →streamlines

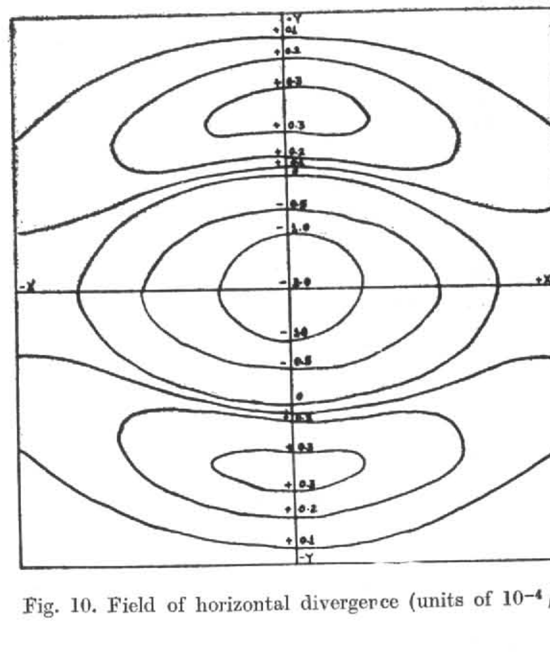

Fig. 10. Field of horizontal divergence (units of 10^{-4} /sec)

where
$$F(\theta) = \left[\left(-\frac{A\cos\theta}{\sigma_1^2} - \frac{B\sin\theta}{\sigma_2^2}\right)^2 + \left(\frac{B\cos\theta}{\sigma_1^2} - \frac{A\sin\theta}{\sigma_2^2}\right)^2\right]$$

$$f(\theta) = \frac{\cos^2\theta}{\sigma_1^2} + \frac{\sin^2\theta}{\sigma_2^2}$$

The investigation of maxima/minima of $|V|^2$ can be done by partially differentiating w.r.t. r and setting $(\partial/\partial r)|V|^2$ as zero. (10)

$$-\frac{\partial}{\partial r}|V|^2 = F(\theta)\{2r - 2r^3 f(\theta)\}\, e^{-r^2 f(\theta)} \qquad (11)$$

$\frac{\partial}{\partial r}|V|^2$ vanishes at $r=0$ and $r=\infty$. It also vanishes at all points defined by $r^2 f(\theta) = \frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} = 1$

We note that $|V|^2$ is zero at $r=0$ and $r=\infty$ and maxima occur at all points on the curve satisfying $\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} = 1$. Hence the speedfield of the model must be characterised by a ring of strong wind. The ring in this case is an ellipse. Fig. 7 gives the isotach field and the elliptical ring of strong wind indicated by a dotted line. It may be noted that the pattern is symmetrical.

Fig. 8 gives isotach field of a cyclone superposed by a uniform easterly current of 10 m/sec. In this case the points of singularity and neutrality are characterised by zero speed. A ring of strong wind encircles the point of singularity. It may be noted that the distribution is assymmetrical with stronger winds in the northern sector as compared to the southern sector.

The effect of superposition merits further investigation. The easterly current of 10 m/sec is relatively weak compared to the speed of wind in a cyclone. If the superposed current is relatively stronger than the wind at the ring of strong wind of a cyclone, the resultant pattern is entirely different. Fig. 9 gives superposition of a uniform easterly current of 50 m/sec on the cyclone whose u and v components are scaled down by a factor of 10. It may be noted that the point of singularity is wiped out and the ring of strong wind is broken. Further, the easterly current is associated with a trough, an isotach maximum and an isotach minimum. We may conclude that the manifestation of a cyclone on the wind field as a spiralling pattern with a ring of strong wind becomes apparent only when the intensity of the cyclone is relatively stronger compared to the supersposing winds.

The divergence field is given by $\nabla . \mathbf{V} = A \nabla^2 \phi$

$$= A \left\{ \frac{x^2}{\sigma_1^4} + \frac{y^2}{\sigma_2^4} - \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right\} \exp \left[- \left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \right] \quad (12)$$

Fig. 10 gives the divergence field and Fig. 11 the profile of divergence along x-axis.

The verticity field is given by

$$(\nabla \times \mathbf{V}) \, . \, \mathbf{k} = - B \nabla^2 \phi = - B \left\{ \frac{x^2}{\sigma_1^4} + \frac{y^2}{\sigma_2^4} - \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right\} \exp \left[- \left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \right] \quad (13)$$

Fig. 12 gives the vorticity field and Fig. 13 the profile of vorticity along x-axis in $\sec^{-1}$.

5. Discussion

The velocity field and its derivative fields associated with the model exhibit close resemblance to those seen in actual tropical cyclones. The vorticity field of the model however differs from that given by Riehl (1950). There is a necessity for fresh determinaiton of the vorticity field experimentally.

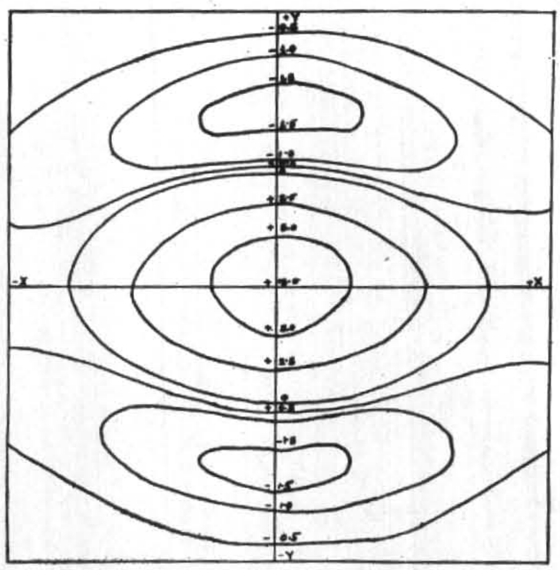

Fig. 11. Field of vertical component of vorticity (units of 10^{-4}/sec)

Representing by $\mathbf{V}$ the velocity vector field o the cyclone and by $\mathbf{V}_E$ the velocity vector field o the superposing current, (sometimes called the embedding current) we note that the point of singularity and the point of neutrality are obtained by solving the equation.

$$\mathbf{V} + \mathbf{V}_E = 0 \quad (14)$$

A graphical method for solving the above equation is as follows. Construct isogon charts for $\mathbf{V}$ and $\mathbf{V}_E$ fields separately. Superposing them on a light table, construct a third chart with lines at every point of which the difference between the isogons is 180°. Similarly construct isotach charts for $\mathbf{V}$ and $\mathbf{V}_E$ fields separately. Superposing them on a light table construct a new chart, *i.e.*, sixth chart with lines at every point of which the difference between $|\mathbf{V}|$ and $|\mathbf{V}_E|$ is zero. Superposing the third on the sixth chart, we obtain points of intersection between lines of the third chart and lines of the sixth chart. The points are points of signularity/points of neutrality.

In the specific case of the model and the unifom easterly current as the superposing current/embedding current. The line at every point of which the difference between the directions is 180°, is the isogon 270°. This is a straight line.

Fig. 14 represents the profile of speed along the isogon 270°, superposed with a line representing the uniform easterly current. The points of intersections S and N give the point of singularity and point of neutrality respectively.

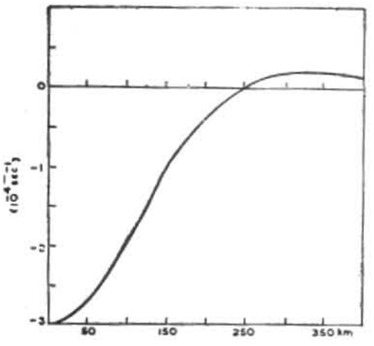

Fig. 12. Profile of horizontal divergence

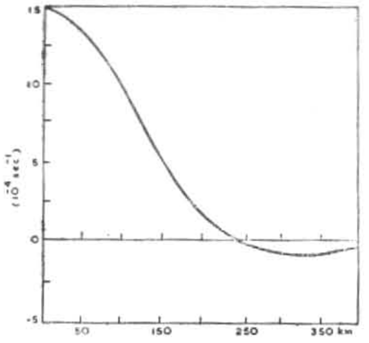

Fig. 13. Profile of vertical component of vorticity

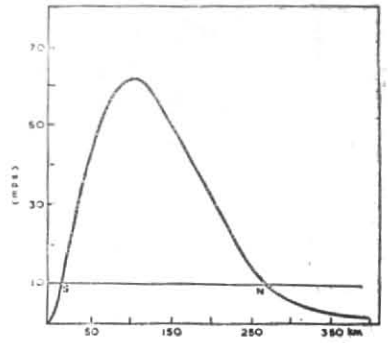

Fig. 14. Speed profile along 270° isogon

It is easily noted that :

(*i*) The distance of separation between S and N increases for smaller values of the easterly current,

(*ii*) the distance of separation between S and N is smaller for large value of the easterly current,

(*iii*) S and N merge into a single point for an easterly current having the speed represented as the maximum on the speed profile for the cyclone along isogon 270° and

(*iv*) There is no point of intersection for superposing uniform currents whose speed is greater than the maximum speed of cyclone along isogon 270° .

Sherman 1953 stresses the importance of the point of singularity and the point of neutrality in the context of huricanes. The approximate relationship according to Sherman is that the hyperbolic point is in the left forward quadrant where left and forward are taken with respect to the direction of the embedding current. The model makes clear the relationship not only in the context of the quadrant but also in the context of the distance separating the point of singularity and the point of neutrality.

If the boundary were finite and the velocity of wind were tangential, the system will still be Helmholtzian and suitable appropriate functions are to be used.

Since the model's features correspond with those observed in reality, it is reasonable to think that the hypothesis is correct. The potential energy density, therefore, must be of the form given in Eq. (1).

Same model can be used for anticyclones also with suitable changes in the sign. If A and B were negative clockwise spiralling outwards pattern will be obtained.

6. Conclusion

A two dimensional mathematical model based on author's theoretical investigation on Helmholtz dynamical system is shown to have close semblance with tropical cyclone.

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