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# Planetary motion calculation method

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ABSTRACT. An accurate method for calculating the unperturbed motion and position of the earth, a planet, or a satellite, moving in an elliptical orbit, as a function of time, is presented. The method is easy to apply and is suitable for use in the modelling of various types of problems in meteorology and geophysics.

## 1. Planetary motion in polar coordinates

Some of the problems encountered in the sciences of meteorology and geophysics require a knowledge of the position of the earth in orbit as a function of time. A suitable method for making this calculation, as well as the calculation of the position of the other planets and the satellites in motion in their orbits, is derived as follows.

If polar coordinates are used, with the origin at the centre of mass of the system, and if the external forces acting on the system are negligible, the angular momentum is given by

$$L = (\mu \cdot r \cdot \dot{\phi}) \cdot r = \text{constant} \qquad (1.1)$$

where $r$ is the radius vector

$\phi$ is the angle in the polar coordinate system

$\dot{\phi}$ is the angular velocity

$\mu$ is the reduced mass of the system; $\mu = m_1 . m_2/(m_1 + m_2)$

$m_1$ and $m_2$ are the masses of bodies 1 and 2 of the system (*e.g.*, the sun and the planet).

Hence,

$$r^2 \cdot (d\phi \,/\, dt) \,/\, 2 = L \,/\, (2.\mu) = \text{constant} \qquad (1.2)$$

which implies that equal areas are swept out by the radius vector, in equal time increments (Kepler's second law).

The orbits of the planets and the satellites of the solar system are ellipses, so that the area swept out by the radius vector during a complete rotation period, $T$, is the area of an ellipse $\pi \cdot a \cdot b$. In Fig. 1, the motion of a planet, (at position P), travelling in an elliptical orbit around the centre of mass of the system, C.M., after an elapsed time, $t$, is shown.

Integrating Eq. (1.2) over a time period, $t$, and then integrating over the time for a complete rotation period, $T$, permits the position of the planet, at any given time, to be represented by

$$\int_0^{\phi} r^2 \cdot d\phi \,/\, 2 = [\,(\pi \cdot a \cdot b) \,/\, T] \cdot \int_0^{t} dt \qquad (1.3)$$

## 2. Parametric representation of the orbit

If the position of the planet, at P, is referred to a coordinate system centered at the geometric centre of the ellipse, at O, the problem may be very considerably simplified using the parametric form for the ellipse, indicating the orbit,

$$x = a \cdot \cos\theta, \quad y = b \cdot \sin\theta \qquad (2.1)$$

where the meaning of the notation is shown in Fig. 1. Also noting that

$$\tan\phi = y \,/\, (x - a \cdot \epsilon) \qquad (2.2)$$

where $\epsilon$ is the eccentricity, (a constant parameter for an ellipse), and then using Eq. (2.1) in Eq. (2.2) results in

$$\phi = \arctan\,[\,(b \,/\, a) \cdot \sin\theta \,/\, (\cos\theta - \epsilon)\,] \qquad (2.3)$$

Differentiating Eq. (2.3) with respect to time, $t$, yields

$$\frac{d\phi}{dt} = a \cdot b \cdot \frac{[\cos\theta \,.\, (\cos\theta - \epsilon) + \sin^2\theta]}{[a^2 . (\cos\theta - \epsilon)^2 + b^2 \,.\, \sin^2\theta]} \,. d\theta/dt \qquad (2.4)$$

A useful relationship for the radius vector, $r$, is given by

$$r^2 = (x - a . \epsilon)^2 + y^2 \qquad (2.5)$$

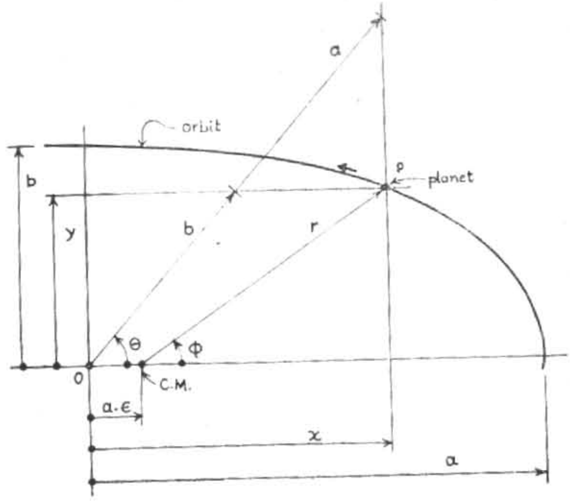


Fig. 1. A planet moving in an elliptical orbit

and in view of Eq. (2.1)

$$r^2 = (a \cdot \cos\theta - a \cdot \epsilon)^2 + b^2 \cdot \sin^2\theta \qquad (2.6)$$

Substituting Eq. (2.6) in Eq. (2.4) and simplifying,

$$d\phi/dt = a \cdot b \cdot [\,(1 - \epsilon \cdot \cos\theta) / r^2\,] \cdot d\theta/dt \qquad (2.7)$$

thus,

$$r^2 \cdot d\phi = a \cdot b \cdot (1 - \epsilon \cdot \cos\theta) \cdot d\theta \qquad (2.8)$$

Eq. (2.8) may now be substituted in Eq. (1.3) to obtain

$$\int_0^\theta a \cdot b \cdot (1 - \epsilon \cdot \cos\theta) \cdot d\theta = 2 \cdot \pi \cdot a \cdot b \cdot t \,/\, T \qquad (2.9)$$

Therefore,

$$\theta - \epsilon \cdot \sin\theta = 2 \cdot \pi \cdot t \,/\, T \qquad (2.10)$$

Eq. (2.10), (known as Kepler's equation), gives the accurate position of a planet in orbit at time, $t$. If $\theta_j$ is an approximation for $\theta$ at step $j$, an improved approximation at step $j+1$ is given by

$$\theta_{j+1} = 2 \cdot \pi \cdot t \,/\, T + \epsilon \cdot \sin\theta_j \qquad (2.11)$$

Hence, by beginning with $\theta_0 = 2 \cdot \pi \cdot t/T$ and repeating the calculation of Eq. (2.11) until $(\theta_{j+1} - \theta_j) < e$, where $e$ is the accuracy desired, a solution can be obtained for Eq. (2.10). The eccentricity, $\epsilon$, is related to the semi-major axis, $a$, and the semi-minor axis, $b$, of the ellipse by

$$\epsilon^2 = 1 - (b/a)^2.$$

When Eq. (2.11) is used for the position of the earth in orbit, $\epsilon = 0 \cdot 016722$, (p. 141, Allen 1973), this method converges rapidly, (*e.g.*, to 7 significant figures of accuracy within about 3 or 4 successive iterations, when interpolating using table 4·6, p. 142, of Abramowitz and Stegun 1964).

The method described is easy to use and is convenient for application when modelling meteorological and geophysical problems which require an accurate knowledge of the earth's position in orbit (*e.g.*, the calculation of the solar zenith angle at various times of the year, or sunrise and sunset time calculations, etc). Eq. (2·11) may also be used for modelling astronomical problems involving the unperturbed orbital positions of the planets and the satellites of the solar system.

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