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Remote sensing of temperature — A numerical experiment

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सार — यदि उचित व्यवरोध का चयन हो जाए तो व्यवरुद्ध रैखिक प्रतिलोमन तकनीक सटीक तथा संख्यात्मक दृष्टि से स्थाई समाधान प्रदान करती है। संख्यात्मक प्रयोगों को दृष्टिगत करते हुए अनुकूल व्यवरोधी परिमापकों के निर्धारण के लिए कुछ चयन नियमों पर यहां विचार विमर्श किया गया है ।

ABSTRACT. Constrained linear inversion technique is shown to yield accurate and numerically stable solution subject to proper choice of constraint. Some selection rules for determining the suitable constraint parameters are discussed in light of the numerical experiment.

1. Introduction

Inference of atmospheric temperature structure is now routinely done by means of satellite-borne radiometers and the advantages and limitations of such a procedure have been discussed widely in the literature. The basic theory of remote sensing of temperature (King 1956, Kaplan 1959) involves measurement of atmospheric radiances in some selected channels of small frequency width. The 15 μ m absorption band of the CO_2 is found to be most suitable for this purpose as CO_2 has a constant mixing ratio in the atmosphere, atleast in the meteorologically important part of it, and interference from other atmospheric constituent is minimum. Change of atmospheric transmission from centre to the wing of the band introduces a measure in term of the layer of atmosphere from which the maximum amount of radiation emerges in a particular channel. Expressed mathematically, the measured radiance, $I(\lambda)$ in a particular channel of central wave length λ is related to the Planck function $B(\lambda, x)$ of the level x in the form

$$I(\lambda) = \int_{x_0}^{x_1} \frac{\partial\tau(\lambda, x)}{\partial x} B(\lambda, x)\, dx \tag{1}$$

where $\tau(\lambda, x)$ is the atmospheric transmittance at level x. The above integral equation is a non-linear one as the Planck function varies also with wavelength. An approximate linear form of the Eqn. (1) can be obtained by replacing the actual Planck functions by those corresponding to a fixed reference wave length $\bar{\lambda}$ through a least square fit : :

$$B(\lambda, x) = a_\lambda \bar{B}(\bar{\lambda}, x) + b_\lambda \tag{2}$$

where a_λ , b_λ are the relevant coefficients. In terms of the modified Planck function and radiance values, Eqn. (1) takes the linear form :

$$a_\lambda^{-1}\left(I(\lambda) - b_\lambda \right) \equiv G(\lambda) = \int_{x_0}^{x_1} K(\lambda, x)\, \bar{B}(x)\, dx \tag{3}$$

where the kernel function $K(\lambda, x)$ is the vertical variation of the transmittance function. Out of the many methods available in the literature for solving a set of Fredholm's equation of first kind the simplest one is the direct inversion method in which the above set of equations is approximated by a matrix eqation :

$$g = A f \tag{4}$$

where g and f are column matrices with number of elements same as the number of channels in the radiometer and A is a square matrix formed out of the kernel function. Determination of values of the Planck function and hence mean temperatures of different atmospheric layers now involves simply the inversion of the matrix A. In practice, such a procedure leads to a physically unacceptable solution with large oscillations yielding even negative values for Planck function. Such oscillations persist even when a better approximation of the integral in Eqn. (3) is made by introducing a larger number of quadrature points and least square solution is obtained by solving the equation

$$A^T g = A^T A f \tag{5}$$

where, A is a rectangular matrix with more columns than rows and A^T is the transpose of A.

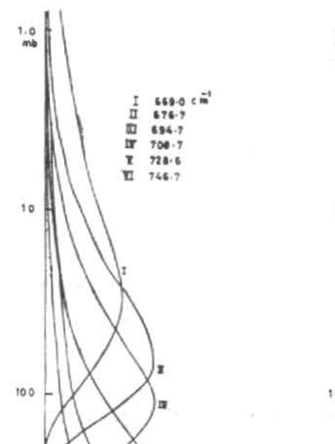

Fig. 1. Variation of kernal function in the vertical

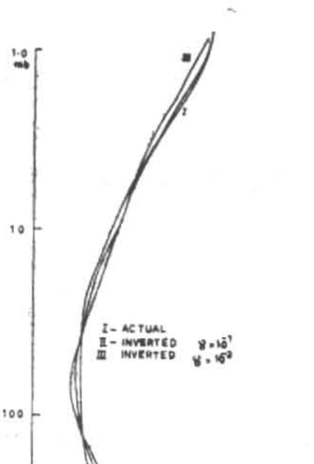

Fig. 2. Actual and inverted temp. profiles

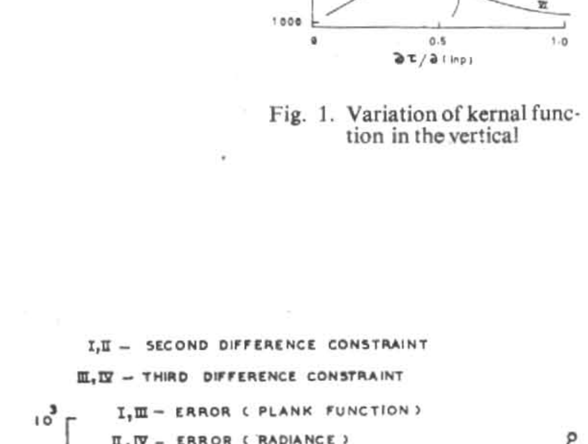

Fig. 3. Variation of error with Gamma

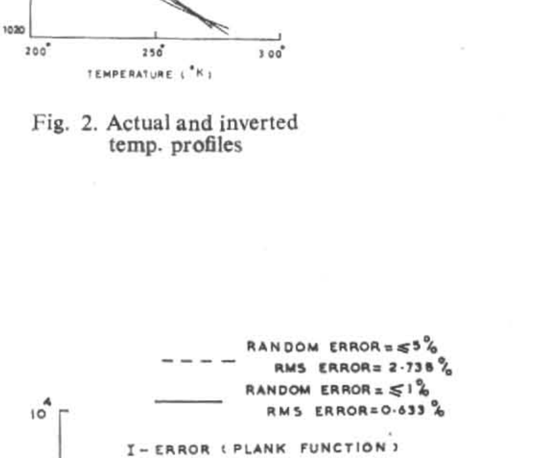

Fig. 4. Result of invesion in presence of random error

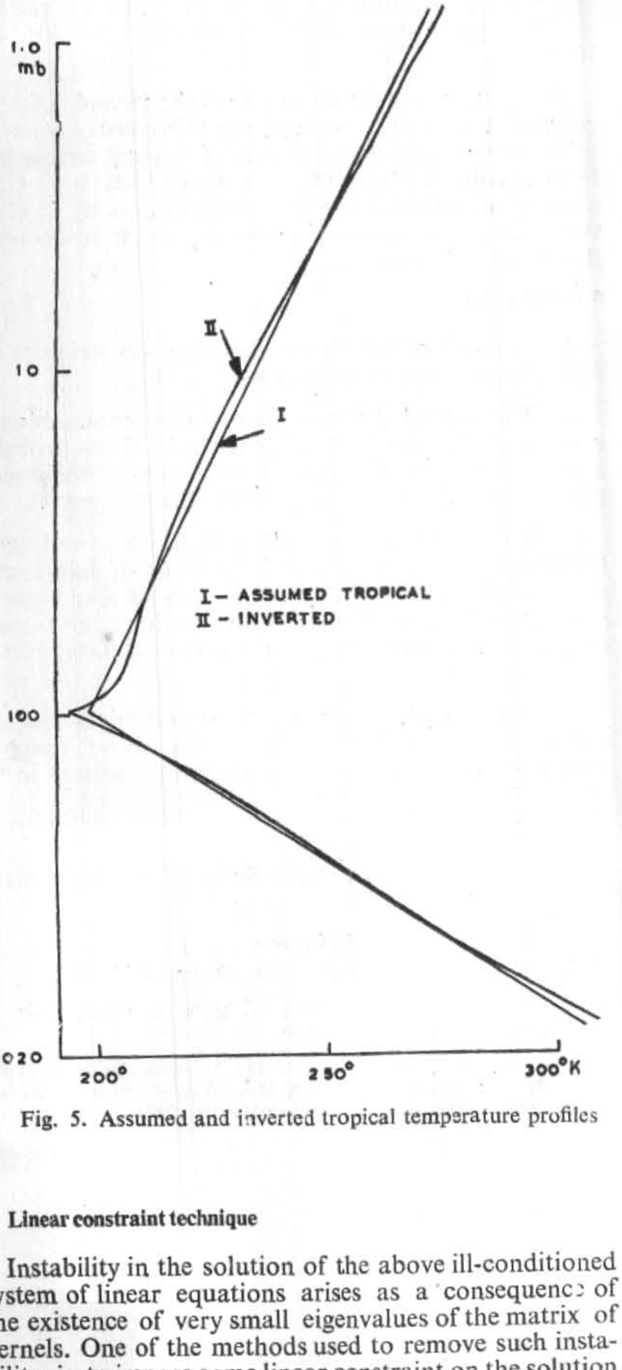

Fig. 5. Assumed and inverted tropical temperature profiles

2. Linear constraint technique

Instability in the solution of the above ill-conditioned system of linear equations arises as a consequence of the existence of very small eigenvalues of the matrix of kernels. One of the methods used to remove such instability is to impose some linear constraint on the solution vector so as to filter out large oscillations. Physically when we seek a solution of the integral Eqn. (3), we essentially seek a set of $\bar{B}(x)$ value such that, for a given matrix of kernels, the values of $G(\lambda)$ is equal to the measured value within limits of experimental error. Obviously, many (probably infinite) sets of $\bar{B}(x)$ values will satisfy the above condition and an externally imposed criteria must be applied to choose the most desirable solution out of this infinite manifold of mathematically plausible solutions. In most problems of *physical interest the solution* appears in form of a smooth function and a linear constraint can be constructed so as to pick out the smoothest out of all possible solutions.

The nth difference matrix D_n can be formed by noting that $D_n f$ is a column matrix whose first n elements are zeros and the remaining elements are :

$$\triangle^n f_i = \triangle^{n-1} f_i - \triangle^{n-1} f_{i-1}$$

where, $\triangle f_i = f_i - f_{i-1}$

The constraint equation :

$$f^T D_n^T D_n f = 0 \tag{6}$$

implies that the sum of the squares of nth differences will vanish—a condition required to push the solution towards any $(n-1)$th order curve. To obtain a smooth solution to the inversion problem Eqns. (5) and (6) are combined through a Lagrangian multiplier and the resultant equation :

$$(A^T A + \gamma D_n^T D_n) f = A^T g \tag{7}$$

is solved. Mathematically, the diagonal and near diagonal elements of the matrix $A^T A$ are incremented by some multiples of the Lagrangian multiplier and this leads to an increase in the eigenvalues of the matrix. The smallest eigenvalue of $A^T A$ gets the largest increment whereas the largest one gets the smallest fractional increment.

Once it is decided to apply the linear constraint technique to the inversion problem, one is faced with the task of choosing the nature of the Lagrangian multiplier suitable to the problem. The latter can be decided upon *aposteriori* by choosing several values of γ and computing the residual g'-g where g' is the radiance vector calculated from Planck functions obtained by solving Eqn. (7). The suitable value of γ is the one for which the residual is comparable to the error in measurement of radiance values. Any higher value of γ leads to a case of over constraint and yields a solution which is oversmooth at the cost of accuracy while a lower value of γ leads to an underconstraint case in which the residual is minimised but the solution is not smooth enough.

The nature of the constraint best suited to the problem at hand can be determined by comparing the results obtained by using different kinds of constraints. In the case where gross features of the solution are already known from past statistics, a constraint can be formulated (Twomey 1977) so as to force the solution vector towards this statistical profile. Such a forcing is, however, not justified when the past data is either meagre or has too large a dispersion. The aim of the present numerical experiment was to explore the possibility of determining the nature of the constraint and the value of the Lagrangian multiplier suitable for inferring temperature from remote sensed radiances without referring to the past statistics.

3. Design of the numerical experiment

The input for the experiment were the temperature profile I of Fig. 2 and the atmospheric transmittance values at selected pressure values corresponding to channels with central wave numbers (in cm^{-1}) as 669.0, 676.7, 694.7, 708.7, 723.6 and 746.7. The atmospheric layer between the surface pressure of 1019.8 mb and the top pressure of 0.8 mb is now subdivided into forty-five layers of equal thickness in the l np scale and the temperature and kernel values are obtained at the logarithmic centre of these layers by fitting a fifth order curve using the Lagrangian interpolation method. The kernel functions are plotted in Fig. 1. The radiance values $I(\lambda)$ are determined for the six channels by using the trapezoidal rule for quadrature to approximately evaluate the integral in Eqn. (1). The vector g is now obtained by a least square fit of Planck function $B(\lambda, x)$ and $B(x)$ where the latter correspond to a fixed wavenumber 700 cm^{-1}. After selecting the suitable form for the constraint matrix $H = D_n^T D_n$, the set of fortyfive simultaneous equations in Eqn. (7) is now solved by Jacobi's method of pivotal condensation and the solution vector f obtained. The radiance vector g' now computed by using the solution vector f in place of planck functions used earlier and the fractional errors $|g'-g|/g$ and $|f-\bar{B}(x)|/|\bar{B}(x)|$ are evaluated for several decades of values of γ.

Next, ten sets each of normally distributed random errors of magnitude not exceeding 1% and 5% are introduced in the vector g and the above procedure repeated.

4. Results

In the present study, second and third difference matrices were used as constraints to recover the Planck functions and hence temperatures. The Lagrangian multiplier is allowed to vary between 10^2 and 10^{-11} and the resulting fractional error in radiance value as well as Planck function are plotted in Fig. 3. It is seen that the fraction error in Planck function attains a broad minimum in the decades $\gamma = 10^{-1}$ to $\gamma = 10^{-3}$ when the second difference matrix is used as constraint. On the other hand the error in radiance for both the second and third difference matrices as constraints achieve a minimum around the value $\gamma = 10^{-6}$ and this is the value of the Lagrangian multiplier one would be forced to choose if there is no other consideration.

In Fig. 4 result of inversion by using the second difference matrix as constraint have been plotted for different values of γ in the case when the input vector g contains normally distributed random error. Comparing the fractional error in Planck functions we find that for second difference matrix as constraint the best result is again obtained for γ values between 10^{-1} and 10^{-3}.

The profile to profile sensitivity of the second difference constraint is tested by applying this constraint on another profile more representative of tropical atmosphere. In this profile the lapse rate is assumed to be 6.5° C/km upto about 100 mb and a rise in temperature by 2° C/km thereafter. The result of constraint linear inversion in this case is plotted in Fig. 5.

5. Conclusions

As a result of the above numerical experiment the following conclusions are reached :

(1) The second difference matrix is more suitable as a constraint than the third difference one. This is probably a reflection of the fact that in most parts of the atmosphere lapse rate of temperature is almost constant.

(2) Best result in terms of Planck function and hence temperature is obtained for the value of Lagrangian multiplier between 10^{-1} and 10^{-3}. Any smaller value of γ leads to the case of underconstraint leading to smaller error in radiance value at the cost of smoothness of solution vector.

(3) The second difference constraint gives rise to stable and acceptable solution in the above range of values of γ even when random error is present in the observed values of radiance.

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