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A non-hydrostatic equatorial atmosphere and the quasi-biennial oscillation*

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सार — भूमध्यरेखीय समतापमण्डल में तरंगों की अन्योन्य-क्रिया के अध्ययन हेतु एक अ-द्रवस्थैतिक, दाबघनत्वी, अपसारी निदर्श का सुझाव दिया गया है । इस अध्ययन से एक प्राचल σ^2 का पता चला है, जो भूमध्यरेखीय वायमण्डल की अ-द्रवस्थैतिकता के विस्तार को प्रदर्शित करता है और अनिवार्यतः केवल पृथ्वी के घूर्णन पर निर्भर है । बहु मापक्रमों की विधि का प्रयोग करके यह दर्शाया गया है कि वायुमण्डल की समस्त भूमध्यरेखीय तरंगों के बीच एक परत वाला दाब घनत्वी निदर्श अकेला ही केलविन तरंगों की अन्योन्यक्रिया को आधार प्रदान कर सकता है । विश्लेषण से प्रकट है कि किस प्रकार अन्योन्यक्रियारत केलविन तरंगें अपनी ऊर्जा रूपान्तरित करते समय मापक्रम T/σ^2 के अनुसार पछुवा हवाएं उत्पन्न करती हैं । यह मापक्रम लगभग 26 मास का है और भूमध्यरेखीय समतापमण्डल में जोनीय सममित पूर्वी एवं पश्चिमी पवन व्यवस्थाओं के नियमित परिवर्तन की अवधि के रूप में जाना जाता है ।

ABSTRACT. A non-hydrostatic, barotropic, divergent model is proposed to study the interaction of the waves in the equatorial stratosphere. The study gives rise to a parameter σ^2 which represents the extent of the non-hydrostaticity of the equatorial atmosphere and essentially depends on the earth's rotation alone. By applying the method of multiple scales, it has been shown that the single layer barotropic model can, among all the equatorial waves in the atmosphere, support the interaction of Kelvin waves alone. The analysis shows how the interacting Kelvin waves transfer energy to generate westerlies on the time scale T/σ^2 which is approximately 26 months that is known to be the period of regular alteration of zonally symmetric easterly and westerly wind regimes in the equatorial stratosphere.

1. Introduction

For prediction of weather on long time scales, meteorologists have sought periodicities in atmospheric motion. Correlation of weather with 11-year solar period, lunar phases and sun spot activity are examples of such periodicities. Recently, attention has been drawn to Quasi-Biennial Oscillations (QBO) and its relation with the monsoon.

Quasi-Biennial Oscillations (QBO) are regular alteration of zonally symmetric easterly and westerly winds† with a period of about 26 months. As pointed out (Holton 1978), successive regimes‡ first appear at 30 km but propagate downwards at the rate of 1 km/month. The downward propagation occurs without loss of amplitude between 30 km and 23 km but rapid attenuation takes place below 23 km. The two regimes are nearly indistinguishable near the troposphere. Another characteristic of the QBO is that the oscillations are found to be symmetric about the equator with a maximum amplitude of 20 m/sec and of half width of about 12° latitude (Reed and Rogers 1961, Varyand and Ebdon 1961).

Large-scale equatorial waves play an important role in producing variations in mean zonal winds associated with QBO. Observations by Hirota (1978) and theoretical studies by others (Holton and Lindzen 1972; Dunkerton 1978; Holton and Lindzen 1968; Lindzen 1967; Lindzen 1970; Lindzen and Holton 1968; Lindzen and Matsuno 1968) indicate that the westerly Kelvin waves and the easterly mixed Rossby-gravity waves are the prominent waves which generate biennial periodicity. Kelvin waves transfer westerly momentum upward thereby providing a source of westerly momentum for setting up of westerly regimes, while mixed Rossby-gravity waves induce a mean meridional circulation having strong horizontal heating, which through the coriolis acceleration, produce an easterly acceleration. It may thus be seen that as a result of influence of Kelvin waves and the mixed Rossby-gravity waves,

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†Associated with the wind oscillation, there is a temperature oscillation that has an amplitude of the order of 3°C at 25 km. As in the case of wind oscillation the phase of the temperature oscillation is *maximum* at the highest levels and progresses downwards. Because the temperature oscillation is considerably small, its structure is known in less detail.

‡The easterlies are slightly stronger than the westerlies so that the zonal wind averaged over an entire period is from the east.

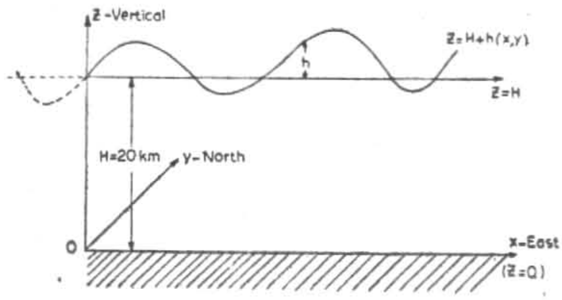

Fig. 1. Shallow water model

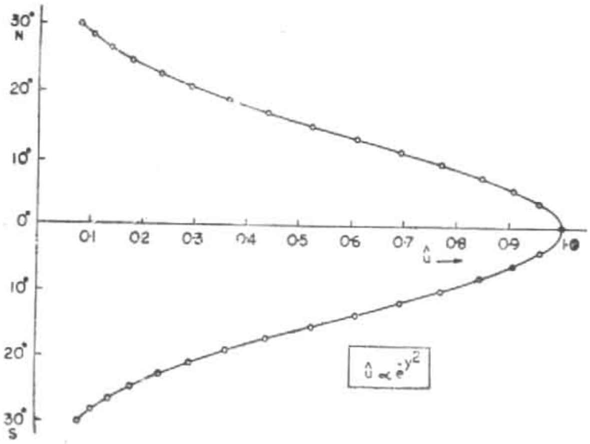

Fig. 2. Zonal profile generated by the interacting Kelvin wave

there is an oscillation in the mean zonal flow, but the alteration from a westerly to an easterly regime has not been well explained.

Thus, any theoretical model to explain QBO should be able to delineate the following characteristic features :

(*i*) Approximate biennial periodicity,

(*ii*) Zonal symmetry about equator and

(*iii*) Downward propagation without loss of amplitude.

As stated above, the earlier studies suggest that the QBO is primarily driven by Kelvin and mixed Rossby-gravity waves but, as far as we are aware, no theoretical study provides adequate information on the mechanism which could explain the near regularity of QBO.

Here, we would confine our attention to the genesis of the westerly component and the periodicity of the QBO. We visualise that the winds in the stratospheric atmosphere are essentially easterlies in character, and the westerly regime is established with a periodicity of 26 months as a result of the self-interaction of Kelvin waves. The change-over to the easterly regime is triggered by the self-interaction of the mixed Rossby-gravity waves.

2. Dynamic model

We consider a divergent barotropic model for studying the interaction between atmospheric waves. We consider a shallow layer of constant depth (Fig. 1) $H\,(=20$ km) in which the fluid is assumed to be inviscid, incompressible and of homogeneous density ρ. Deformations h of the free surface are assumed to be sufficiently small so that free surface variations can be neglected, when horizontal divergence is incorporated into the model. The important dynamic effects of the earth's rotation and sphericity are taken into account by introducing the coriolis force vector :

$$2\Omega\,(w\cos\phi - v\sin\phi,\; u\sin\phi - u\cos\phi)$$

in the equations of motion. A cartesian coordinate system (O, XYZ) is used and the components of the vector are, respectively, in the zonal direction (X), meridional direction (Y) and the Z direction (which is antiparallel to the gravity vector). Ω is the earth's angular speed, ϕ the latitude and (u, v, w) are the components of velocity along OX, OY and OZ.

In this local frame of reference, the equations of motion and of the mass conservation can be written in the form :

$$u_t + uu_x + vu_y + wu_z - 2\,\Omega v\sin\phi + 2\Omega w\cos\phi = -\frac{1}{\rho}\,p_x \tag{1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega\,u\sin\phi = -\frac{1}{\rho}\,p_y \tag{1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega\,u\cos\phi = -g - \frac{1}{\rho}\,p_z \tag{1c}$$

$$u_x + v_y + w_z = 0 \tag{1d}$$

where p is the dynamical pressure and g is the acceleration due to gravity. The latitude ϕ can be expressed in terms of the meridional coordinate by approximating $2\Omega\sin\phi$ by

$$2\,\Omega\phi = 2\Omega\,\frac{y}{a} = \beta\,y \tag{2}$$

where $\beta = 2\Omega/a$ (where a is the earth's radius $\sim 6.37\times 10^6$ m) is the Rossby parameter. The approximation is valid because we are confining our attention to the equatorial atmosphere. Likewise

$$\cos\phi \sim 1 \tag{3}$$

The magnitudes are : $\Omega = .7292\times 10^{-4}\,\mathrm{s}^{-1}$, $\beta = +\,2.2894\times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$).

3. Scale analysis

Notice that $c=(gH)^{1/2}$ represents the speed of the shallow water gravity waves. Taking $H=20\times 10^3$ m, $g=10$ ms^{-2}, we find that $c \simeq 4.472\times 10^2$ m/s, but as we are interested in the slower moving systems, we introduce the characteristic velocity $U{\sim}10$ m/s which is the typical speed corresponding to the phase speed of the westward moving Rossby waves. The characteristic features of the physical model are incorporated by referring the horizontal length scale to $L=\sqrt{(c/\beta)}$ and the time to $T=1/\sqrt{(\beta c)}$. Accordingly, we get

$$L{\sim}10^6 \text{ m and } T{\sim}2.745 \text{ hours.} \tag{4}$$

The characteristic vertical speed W is obtained from the continuity equation 1 (d) :

$$W \sim U \frac{H}{L} \sim 10^{-2} \text{ m/s.}$$

At this stage, it would be appropriate to examine the relative magnitude of the coriolis terms in 1 (a) :

$$| 2 \Omega w \cos \phi | << | 2 \Omega v \sin \phi |$$

In view of this condition, $2\Omega w \cos \phi$ is, traditionally, dropped in meteorological studies. However, at the equator, $2 \Omega v \sin \phi$ vanishes and if the effect of earth's rotation is to be taken into account, the term $2\Omega w \cos \phi$ cannot be ignored in 1(a) for studies in the equatorial atmosphere. This implies (from the energy balance principle) that the term $2\Omega u \cos \phi$ will also have to be retained in 1(c). From this analysis it follows that the so-called "hydrostatic approximation" would no longer be valid in such a situation. Let us, therefore, examine the z-component of the momentum Eqn. 1(c) first. The instantaneous acceleration term, convective terms, coriolis term, pressure gradient term and the earth's gravitational terms in 1(c) are, respectively, of the order :

$$W/T \,(\sim 10^{-6}), \quad UW/L \sim W^2/H \,(\sim 10^{-7}),$$

$$2 \Omega U (\sim 10^{-3}), \; (\triangle p)_{\text{vertical}} / \rho H (\sim 10) \text{ and } g (\sim 10).$$

It follows that in the tropical atmosphere, 1(c) reduces to

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = - g + 2 \Omega u \tag{5}$$

Since $z = H + h$ is a free surface along which pressure will be constant, we have (*see* Fig.1) :

$$0 = (\delta p)_{z=H+h} = \frac{\partial p}{\partial 1} \delta 1 + \frac{\partial p}{\partial z} \delta z$$

where $\delta 1$ represents variation in the horizontal direction.

It follows that

$$\frac{\partial p}{\partial x} = - \frac{\partial p}{\partial z} \frac{\partial z}{\partial x}$$

and

$$\frac{\partial p}{\partial y} = - \frac{\partial p}{\partial z} \frac{\partial z}{\partial y}$$

$$\therefore - \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\rho} \frac{\partial z}{\partial x} \frac{\partial p}{\partial z} = \frac{\partial h}{\partial x} \left[-g + 2 \Omega u \right] = \frac{\partial \phi}{\partial x} \left[-1 + \frac{2 \Omega u}{g} \right]$$

Similarly,

$$- \frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial \phi}{\partial y} \left[-1 + \frac{2 \Omega u}{g} \right],$$

where $\phi = gh$ is the geopotential whose characteristic scale is cU. In non-dimensional form, 1(a) and (b) accordingly, become

$$u_t + \epsilon \left[uu_x + vu_y + wu_z \right] - vy + \frac{\sigma^2}{\epsilon} w = \phi_x \left[-1 + \sigma^2 u \right] \tag{6a}$$

$$v_t + \epsilon [uv_x + vv_y + wv_z] + uy = \phi_y [-1 + \sigma^2 u] \tag{6b}$$

where,

$$\epsilon = \frac{U}{c} \sim 10^{-2} \; (= U/\beta L^2) \tag{7}$$

which can be interpreted as the equatorial equivalent of the conventional Rossby number and

$$\sigma^2 = \frac{2 \Omega U}{g} \sim 2 \Omega \sim 10^{-4} \tag{8}$$

which represents the extent of the non-hydrostaticity of the equatorial atmosphere. Notice that this parameter essentially depends directly on the earth's rotation.

We now integrate the non-dimensional form of the continuity equation which yields

$$w + \int_0^{1+\epsilon\phi} (u_x + v_y) \, dz = 0 \tag{9}$$

(Notice that the upper limit $z = H + h$ becomes $1 + \frac{h}{H}$ in dimensionless form which is equal to $1 + \frac{cU\phi}{c^2} = 1 + \epsilon\phi$).

Since $w = \frac{dh}{dt}$ can be written in the non-dimensional form as $w = \phi_t + (u \phi_x + v \phi_y)$, we have, from Eqn. (9) :

$$\phi_t + (u\phi_x + v\phi_y) + \int_0^{1+\epsilon\phi} (u_x + v_y) \, dz = 0 \tag{9a}$$

The model chosen here would incorporate the special featutres of the QBO like the periodicity and the development of the westerly regime due to the interaction of the waves in the equatorial atmosphere in the stratosphere. Other characteristic features like the vertical propagation of atmospheric waves and upward transfer of momentum due to their interaction cannot be exhibited by this single layer model. Likewise, establishment of the easterly regime due to the interaction of Rossby-gravity waves (which induces a mean meridional circulation having strong horizontal heating) cannot be accounted for by this barotropic model. In our subsequent study, it is proposed to incorporate these special features as well as by a two-level baroclinic model. The model chosen here is well suited for determining the dynamics of the interacting atmospheric waves in the monsoon region.

The nonlinearity of the system implies that the superposition of the waves is not valid. Instead the vorticity of one wave is advected by the velocity of the other and the waves are entrapped by nonlinearity to interact with one another for the energy exchange among themselves.

It may be seen that the zeroth order approximation (obtained by letting ϵ, σ^2 and $\sigma^2 / \epsilon \rightarrow 0$) contains the important equatorial waves — Kelvin, mixed Rossby-gravity, Rossby and eastward and westward propagating inertia-gravity waves.

The problem posed above corresponds to the large cumulative effects represented by the nonlinear advective terms and other small perturbation terms. A uniformly valid asymptotic solution of the system can be found by introducing slow variables to avoid the secular terms. At this stage, it would be appropriate to write:

$$\epsilon = \alpha \sigma \tag{10}$$

where $\alpha = 0(1)$.

The relevant long scales representing slow changes of the system in this study are :

$$X_1 = \sigma x, \; X_2 = \sigma^2 x, \; \ldots\ldots\ldots$$
$$T_1 = \sigma t, \; T_2 = \sigma^2 t, \; \ldots\ldots\ldots \tag{11 a}$$

Accordingly,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \sigma \frac{\partial}{\partial X_1} + \sigma^2 \frac{\partial}{\partial X_2} + \ldots\ldots$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \sigma \frac{\partial}{\partial T_1} + \sigma^2 \frac{\partial}{\partial T_2} + \ldots\ldots \tag{11 b}$$

Before we proceed further with the mathematical analysis, it would be interesting to physically identify the various time scales in terms of the meteorogical processes.

Notice that $T \sim 10^4$ seconds ($\sim$3 hours) which corresponds to meso-scale (applicable to processes of thunderstorms, cumulus thunder, lake breeze, sea breeze etc). The long scale $T_1 (= \sigma t)$ has the characteristic time scale $\sim$ 9 days which is synoptic in character and is applicable to large scale meteorological processes-Rossby waves, monsoon depressions, cyclones etc. $T_2 (= \sigma^2 t)$ is of the order of 26 months which corresponds to the period of QBO that are of significance in the study of Kelvin and mixed Rossby-gravity waves.

4. Analysis

We now set up perturbation expansion of the form :

$$u(\sigma) = u_0 + \sigma u_1 + \sigma^2 u_2 + \ldots\ldots\ldots \tag{12}$$

and similar expansions for v and ϕ, where u_i, v_i, ϕ_i; $i = 0, 1, 2, \ldots.$ depend on the space and time variables (short as well as long scales). Since ϕ is independent of z, it follows from (6a) and (6b) that each successive term $u_0, u_1, u_2, \ldots; v_0, v_1, v_2, \ldots$ of the perturbation expansions will be independent of z. This implies that u and v as well as ϕ depend only on $x, y, t, X_1, T_1, X_2, T_2, \ldots\ldots$ etc. Hence Eqns. (6) reduce to the following form :

$$u_t + \alpha\sigma (uu_x + vu_y) - yv + \frac{\sigma}{\alpha} w = \phi_x(-1 + \sigma^2 u)$$
$$v_t + \alpha\sigma (u v_x + vv_y) + yu = \phi_y(-1 + \sigma^2 u)$$
$$\phi_t + \alpha\sigma (u\phi_x + v\phi_y) + (1 + \alpha\sigma\phi)(u_x + v_y) = 0 \tag{13}$$

Substituting (11 b) and (12) in the above system of equations, the problems of different orders are given by :

$0(1)$:

$$u_{0,t} - yv_0 + \phi_{0,x} = 0$$
$$v_{0,t} + yu_0 + \phi_{0,y} = 0$$
$$\phi_{0,t} + u_{0,x} + v_{0,y} = 0 \tag{14}$$

$0(\sigma)$:

$$u_{1,t} - yv_1 + \phi_{1,x} = -u_{0,T_1} - \frac{1}{\alpha}\phi_{0,t} - $$
$$- \phi_{0,X_1} - \alpha(u_0 u_{0,x} + v_0 v_{0,y})$$
$$v_{1,t} + yu_1 + \phi_{1,y} = -v_{0,T_1} - \alpha(u_0 v_{0,x} + v_0 v_{0,y})$$
$$\phi_{1,t} + u_{1,x} + v_{1,y} = - u_{0,X_1} - \phi_{0,T_2} -$$
$$- \alpha\left[(u_0\phi_0)_x + (v_0\phi_0)_y\right] \tag{15}$$

$0(\sigma^2)$:

$$u_{2,t} - yv_2 + \phi_{2,x} = - u_{1,T_1} - u_{0,T_2} -$$
$$- \frac{1}{\alpha}(\phi_{1,t} + \phi_{0,T_1})$$
$$- v_0\phi_{0,y} - \phi_{1,X_1} - \phi_{0,X_2} - \alpha\left[(v_0 u_1)_x + u_0 u_{0,X_1} + \right.$$
$$\left. + v_0 u_{1,y} + v_1 u_{0,y}\right]$$
$$v_{2,t} + yu_2 + \phi_{2,y} = - v_{1,T_1} - v_{0,T_2} - u_0\phi_{0,y}$$
$$- \alpha[u_0 v_{1,x} + u_0 v_{0,X_1} + u_1 v_{0,x} + (v_1 v_0)_y]$$
$$\phi_{2,t} + u_{2,x} + v_{2,y} = u_{1,X_1} - u_{0,X_2} - \phi_{1,T_1} - \phi_{0,T_2}$$
$$- \alpha[(u_1\phi_0)_x + (u_0\phi_0)_{X_1} + (u_0\phi_1)_x + (v_1\phi_0)_y + (v_0\phi_1)_y] \tag{16}$$

The solution of 0 (1) problem is given by [Matsuno 1966]

$$u_0 = -\left[\frac{(\lambda+k)}{2}\psi_{n+1} + n(\lambda - k)\psi_{n-1}\right](A\, e^{i\theta} + A^* e^{-i\theta})$$
$$v_0 = i(\lambda^2 - k^2)\psi_n (Ae^{i\theta} - A^* e^{-i\theta})$$
$$\phi_0 = -\left[\frac{(\lambda+k)}{2}\psi_{n+1} - n(\lambda - k)\psi_{n-1}\right] \times$$
$$\times (Ae^{i\theta} + A^* e^{-i\theta}) \tag{17}$$

where,

$$\lambda^2 - k^2 - k/\lambda = 2n + 1, \; \theta = kx - \lambda t,$$
$$A = A(X_1, T_1, X_2, T_2 \ldots.), \; \psi_n = H_n(y)\, e^{-y^2/2}$$

H_n being the Hermite polynomial of order n.

For Kelvin wave solution, $n = -1$ and $\lambda = k = k_0$ Hence, from (17) we find that

$$u_0 = \phi_0 = e^{-y^2/2}(Ae^{i\theta_0} + A^* e^{-i\theta_0}), \; v_0 = 0 \tag{18}$$

where $\theta_0 = k_0(x - t)$.

5. 0 (σ) solution

Eliminating u_1, and ϕ_1 from Eqns. (15) we obtain

$$\mathcal{L}(v_1) \equiv v_{1,ttt} - v_{1,xxt} + y^2 v_{1,t} - v_{1,yyt} - v_{1,x} = -v_{0,ttT_1} + v_{0,xxT_1} + u_{0,ytX_1} - \phi_{0,ytT_1} - u_{0,xyT_1} - \phi_{0,xyX_1}$$
$$+ y(-u_{0,xX_1} - \phi_{0,xT_1} + u_{0,tT_1} + \phi_{0,tX_1}) + \frac{1}{\alpha}(-\phi_{0,xyt} + y\phi_{0,tt})$$
$$+ \alpha\{-(u_0 v_{0,x} + v_0 v_{0,y})_{tt} + (u_0 v_{0,x} + v_0 v_{0,y})_{xx} - y[(u_0\phi_0)_{xx} + (v_0\phi_0)_{xy} + (u_0 u_{0,x})_t + (v_0 u_{0,y})_t]$$
$$+ (u_0\phi_0)_{xyt} + (v_0\phi_0)_{yyt} - (u_0 u_{0,x})_{xy} - (v_0 u_{0,y})_{xy}\} \quad (19)$$

Letting $v_1 = v\,(y)\, e^{i\theta}$ the left side of (19) becomes

$$\mathcal{L}(v_1) \equiv \{v''(y) + [(\lambda^2 - k^2 - k/\lambda) - y^2]\, v(y)\}\, e^{i\theta} \quad (20)$$

Substituting (17) in the right side of (19), the linear terms reduce to

$$(\lambda^2 - k^2)\{i(2\lambda^2 + k/\lambda)\psi_n(A_{T_1}\, e^{i\theta} - A^*_{T_1} e^{-i\theta}) + i(1 + 2\lambda k)\, \psi_n(A_{X_1}\, e^{i\theta} - A^*_{X_1}\, e^{-i\theta}) +$$
$$+ \frac{\lambda}{\alpha}\left[\tfrac{1}{4}\psi_{n+2} + (\lambda k + \tfrac{1}{2})\psi_n - n(n-1)\psi_{n-2}\right] \times (Ae^{i\theta} + A^* e^{-i\theta})\} \quad (21)$$

The contribution of the nonlinear terms of the right side of (19) is given in Appendix 1. An examination of (21) shows that the forced modes are in resonance with the natural modes of the system and should therefore be avoided on physical grounds. This implies that the only acceptable solution would correspond to $\lambda = k$ ($\lambda = -k$ gives a divergent solution) and $n = -1$. That is, in the nonhydrostatic barotropic equatorial atmosphere, only the self-interaction of Kelvin waves is possible. We accordingly obtain :

$$[v'' - (1 + y^2)\, v]\, e^{i\theta_0} = -6\alpha k_0^4 y e^{-y^2} \times (A^2 e^{2i\theta_0} - A^{2*} e^{-2i\theta_0}) \quad (22)$$

whose solution is discussed in Appendix 2.

As we are primarily interested in determining the zonal flow arising due to the interaction among atmospheric waves, the non-oscillatory components described below would be of fundamental interest for our present study.

$0(\sigma)$ non-oscillatory part :

$$y\hat{u}_1 + \hat{\phi}_{1,y} = 0 \quad (23a)$$

$0(\sigma^2)$ non-oscillatory part :

$$+ y\hat{v}_2 = + \hat{u}_{1,T_1} + \hat{\phi}_{1,X_1} + \alpha[\wedge(u_0 u_{0,X_1}) + \wedge(v_1 u_{0,y})] \quad (23b)$$

$$-\hat{v}_{2,y} = \hat{u}_{1,X_1} + \hat{\phi}_{1,T_1} + \alpha[\wedge(u_0\phi_0)_{X_1} + \wedge(v_1\phi_0)_y] \quad (23c)$$

where $\wedge$ denotes the nonoscillatory part of the quantity. Eqns. (23) form a coupled system for the unknowns $\hat{v}_2$, $\hat{u}_1$, $\hat{\phi}_1$ to yield zonal flow on temporal and spatial scales.

There arise two special cases corresponding to the amplitudes of the interacting waves being (1) independent of the planetary length scales X_1, X_2, ...and (2) independent of the temporal scales T_1, T_2, ...

6. Results

6.1. *Case study* 1

When the waves remain discrete all along, *i.e.*, their amplitudes depend only on the tamporal scales T_1, T_2 ......, the zonal Eqns. (23) reduce to

$$\mathcal{L}_1(\hat{v}_2) \equiv \hat{v}_{2,yy} - y^2\hat{v}_2 = \alpha y(8y^2 - 9)\, e^{-3y^2/2} \times |A|^2\,(A - A^*). \quad (24)$$

The operator $\mathcal{L}_1$ is a special case of the operator $[(d^2/dy^2) + (2n + 1 - y^2)]$ which is known to exhibit unrealistic divergent solution for non-integral values of n. Hence, Eqn. (24) does not provide any realistic solution for $\hat{u}_1$. Thus local features of interactions among the Kelvin waves of discrete spectrum, having amplitudes independent of the planetary length scales X_1, X_2 ,....do not generate any zonal current on long time scales T_1, T_2, ... This confirms one of the conclusions of Loesch (1977) about the incapability of the interacting atmospheric waves of exciting any zonal flow on temporal scales T_1, T_2.....

6.2. *Case study* 2

The continuity of wave spectra and invariance of amplitudes of the interacting waves on the temporal scales T_1, T_2...., lead to the simplification of Eqns. (23) to the form :

$$y\hat{v}_2 = \hat{\phi}_{1,X_1} + \alpha[\wedge(u_0 u_{0,X_1}) + \wedge(v_1 u_{0,y})] \quad 25(a)$$

$$-\hat{v}_{2,y} = \hat{u}_{1,X_1} + \alpha[\wedge(u_0\phi_0)_{X_1} + \wedge(v_1\phi_0)_y] \quad 25(b)$$

where u_0, ϕ_0 are the Kelvin wave solutions given by (18) and v_1 is the solution of (22). Differentiating 25(a) with respect to y and substituting 25(b) for $v_{2,y}$, we get

$$\hat{\phi}_1 = \alpha e^{-y^2}[|A(0,0)|^2 - |A(X_1, X_2)|^2]$$
$$\hat{u}_1 = 2\alpha e^{-y^2}[|A(0,0)|^2 - |A(X_1, X_2)|^2] \quad (26)$$

which shows how energy is transferred from the interacting Kelvin waves for the generation of the zonal flow (Fig. 2).

The study shows that the non-hydrostatic condition imposes the restriction that out of all the important atmospheric waves in the equatorial region, only the self-interaction of Kelvin waves is possible and the remaining waves give rise to resonance of the forced modes with the natural modes and, therefore, should be avoided on physical grounds. Evidently, baroclinicity of the equatorial atmosphere would account for the interaction, if any, of other atmospheric waves in the monsoon region. It is shown that the Kelvin wave triads cannot be in resonant interaction to excite the zonal flow and that the self-interaction of Kelvin waves consequent to their entrapping by convective non-linearities on planetary scale to the transfer of energy for the generation of the westerlies. The characteristic scale of this self-resonance corresponds to the periodicity of the QBO.

As consideration of easterly regime induced by the mixed Rossby-gravity modes needs inclusion of temperature effect, the above study would not be appropriate from the point of view of this aspect.

The self-resonance provides excitation of westerly zonal flow, *i.e.,* Kelvin modes transfer westerly momentum which has also been indicated by earlier investigators (Holton and Lindzen 1972). The interesting point of the study is that the maximum transfer would take place during resonance. This self-resonance has a period of T/σ^2 which corresponds to a time scale of 26 months which is the periodicity of the QBO.

The significant conclusions from this study are that the self-interaction of the equatorial Kelvin waves accounts for the excitation of the zonal westerlies and the period of intense excitation is of the order of 26 months which is the period of the QBO.

Two interesting points are noteworthy in this connection : (*i*) the non-hydrostaticicity of the atmosphere has been shown to be linked with the periodicity of the QBO. Since the non-hydrostaticity of the atmosphere and the QBO are the characteristic features of the equatorial atmosphere, this correlation, although surprising, is understandable; (*ii*) the periodicity of the QBO is known to be approximately 26 months. Here, the 26-month periodicity arises due to the resonance occurring at $0(\sigma^2)$. The corresponding time scale is :

$$\frac{T}{\sigma^2} = \frac{\beta^{-\frac{1}{2}}\, g^{-\frac{1}{4}}\, H^{-\frac{1}{4}}}{2\,\Omega}$$

where the non-hydrostaticity parameter σ^2 of the equatorial atmosphere essentially depends on the earth's rotation alone, and the characteristic time scale T (which is of the meso-scale order) $= \dfrac{1}{\beta^{\frac{1}{2}}\, g^{\frac{1}{4}}\, H^{\frac{1}{4}}}$. The periodicity of the QBO will, therefore, depend on the depth H of the atmosphere above the earth's surface in the equatorial stratosphere. The following table gives the period of the QBO *vs* H.

Depth of the atmosphere H (km)	Period of the QBO T/σ^2 (months)
15	28
20	26
25	24
30	23

This is, in accordance with the observational data according to which the periodicity of the QBO is known to vary with the depth of the atmosphere from 22 months to 30 months or so (Newell *et al.* 1974).

The other characteristic features like vertical transportation of the momentum as observed by (Reed and Rogers 1961, Varyand and Ebdon 1961) etc and the development of the easterly component of the QBO, cannot be explained, as pointed out earlier, by this single layer barotropic model. Our subsequent study would throw light on these aspects of the QBO.

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Appendix 1

The non-linear inhomogeneous terms of (19) are given by

$$\alpha\Big\{[(\lambda^2-k^2)\{(\lambda^2-k^2)(\lambda^2+2k^2+3\lambda k+3)+\frac{nk}{2}(4k-5\lambda)$$

$$+\frac{3k(\lambda+k)}{4}(\lambda^2-k^2-k/\lambda+5)\}]\,\psi_n\,\psi_{n+1}$$

$$+(\lambda^2-k^2)\{4kn(\lambda-k)(\lambda^2-k^2)(1-\lambda-k)-$$

$$-2n(n+1)(\lambda^2+\frac{9}{2}\lambda k+k^2)+n^2(\lambda-k)(\lambda+3k)-$$

$$-nk(\lambda+k)+\frac{k}{4}\left(\lambda-k-\frac{1}{\lambda}\right)^2(5\lambda-4k)\}]\,\psi_n\,\psi_{n-1}$$

$$+\left[-(\lambda^2-k^2)(\lambda+k)\frac{(2\lambda-k)}{4}-\frac{3\lambda k}{4}(\lambda+k)^2\right]\psi_{n+1}\,\psi_{n+2}$$

$$+\left[-2n^2(n-1)(\lambda-k)\{(\lambda^2-k^2)(4\lambda-k)+3\lambda k(\lambda-k)\}\right]\psi_{n-1}\,\psi_{n-2}$$

$$+\left[\frac{n}{2}(\lambda^2-k^2)(2\lambda^2-k^2)+4\lambda k\right]\psi_{n-1}\,\psi_{n+2}$$

$$+\left[n(n-1)k^2(\lambda^2-k^2)\right]\psi_{n+1}\,\psi_{n-2}$$

$$+\left[-(\lambda^2-k^2)\frac{(\lambda+k)^2}{4}\right]\psi_n\,\psi_{n+3}$$

$$+\left[2n(n-1)(n-2)(\lambda-k)^2(\lambda^2-k^2)\right]\psi_n\,\psi_{n-3}\Big\}(A^2e^{2i\theta}-A^{*2}e^{-2i\theta}) \qquad \text{(A1)}$$

Appendix 2

Solution of the equation

$$v'' - (1 + y^2)v = ye^{-y^2} \quad \text{(B1)}$$

With the boundary conditions

$$v(\pm\infty) = 0. \quad \text{(B2)}$$

The solution of the homogeneous equation

$$v'' - (1 + y^2)v = 0 \quad \text{(B3)}$$

is sought in series form as follows :

$$v = y^r \sum_{k=0}^{\infty} g_{k,r}\, y^k \;;\; g_{0,r} = 0 \quad \text{(B4)}$$

putting (B 3) in Frobenins from, substituting (B 4) for v and equating the coefficients of y^k on both sides of the identity we get $r = 0, 1$, and

$$g_{k,r} = \frac{(g_{k-2,r} + g_{k-4,r})}{(r+k)(r+k-1)},\; g_{k-1},\; r = 0,$$

$$k = 2, 4, \quad \ldots \quad \text{(B5)}$$

where,

$$g_{k-t,r} = 0 \text{ for } k<t \text{ and } g_{0,r} = 1.$$

The two linearly independent solution of (B 3) are then given by

$$v_0 = \sum_{k=0}^{\infty} g_{2k,0}\, y^{2k}, \quad v_1 = \sum_{k=0}^{\infty} g_{2k,1}\, y^{2k+1} \quad \text{(B6)}$$

Particular solution of (B 1)

Let $v_p = G(y)e^{-y^2}$ be a particular solution of (B 1). Then $G(y)$ satisfies the equation

$$G''(y) - 4y\,G'(y) + 3\,(y^2 - 1)\,G(y) = y \quad \text{(B7)}$$

Let $G = \sum_{k=0}^{\infty} A_k y^k$ be a particular solution of (B 7). Then substituting the above expression in (B 7), the coefficient A_k are found to be

$$A_0 = A_1 = A_2 = 0,\; A_3 = 1/6,$$

$$A_k = \frac{(4k-5)\,A_{k-2} - 3A_{k-4}}{k(k-1)},$$

$$A_{k-1} = 0,\; k = 5, 7, 9, \ldots\ldots \quad \text{(B8)}$$

The complete solution of (B 1) is then given by

$$v = c_0 v_0 + c_1 v_1 + v_p \quad \text{(B9)}$$

where c_0, c_1 are arbitrary constants. Utilising the boundary condition (B 2) we find that

$$c_0 = 0 \text{ and } c_1 = \lim_{y\to\infty} - v_p(y)/v_1(y), \quad \text{(B10)}$$

so that

$$v = c_1 v_1 + v_p \quad \text{(B11)}$$