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Precipitation network design for mountainous catchment

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सार — पर्वतीय क्षेत्र में वर्षण की स्थानिक परिवर्तनशीलता अधिक होती है और s दूरी के दो स्थानों में अभिलेखित वर्षण श्रृंखलाओं के मध्य सहसंबंध, $[r(s)]$ मैदानी क्षेत्रों की अपेक्षा s के साथ अधिक तेजी से कम होता है। इस प्रकार क्षेत्रीय वर्षण के आकलन के लिए पर्वतों में सघन केन्द्र संजाल की आवश्यकता है।

इस शोधपत्र में क्षेत्रीय वर्षण के अन्तर का पता लगाने के लिए सह-संबंध पर विचार विमर्श किया गया है और (क) बाढ़ पूर्वानुमान घटनाओं, (ख) दीर्घविधि जलवायु विज्ञान उद्देश्यों के संबंध में केन्द्र संजाल सघनता के मूल्याकन के लिए एक प्रक्रिया की रूपरेखा प्रस्तुत की गई है।

पश्चिमी हिमालय में व्यास जलग्रहण (क्षेत्र–12509 कि. मी. वर्ग) के लिए प्रस्तावित योजना को चित्रित किया गया है।

ABSTRACT. The precipitation in mountainous region has higher spatial variability and the correlation $r(s)$ between precipitation series recorded at two stations distant 's' apart decreases with s at faster rate than in plain region. As such, a denser network would be desirable in mountains for the estimation of areal precipitation.

In this paper, the correlation approach to derive the variance of areal precipitation has been discussed and a procedure is outlined for evaluation of network density in respect of (a) flood forecasting events, (b) long term climatological purposes.

The scheme suggested has been illustrated for Beas catchment (area: 12509 km²) in Western Himalayas.

1. Introduction

The annual/seasonal precipitation over a mountainous catchment is generally characterised by two factors, namely,

(*i*) higher spatial variability, which is due mainly to abruptly large and significant variations in terrain and

(*ii*) the faster convergence of spatial correlation structure to an insignificant value (Ramanathan *et al.* 1981).

The number of precipitation gauges required to estimate areal mean with a specified precision usually depends upon these two factors. As such, under similar conditions, a mountainous basin will require higher density of network than the plain watersheds.

In this paper, a technique for the determination of optimum network density has been evolved by deriving variance $V(p_A)$ of areal mean p_A as a function of temporal and spatial variations. This has been discussed in detail in section 3.

The correlation approach between precipitations of two stations was first used by Hershfield (1965) to design the proper spacing between stations and subsequently by Kagan (1966) to evaluate the errors in interpolation of precipitation values. Later, Iturbe *et al.* (1974) and Eagleson (1980) used similar concept and also introduced temporal and spatial reduction factors to the variance of areal mean. Ramanathan *et al.* (1981) have given a review of the work done in network design till 1977.

The technique evolved in this paper based on a similar approach is illustrated for the Beas catchment. It covers an area of 12509 sq. km between the longitudes 75°54′E-77°54′E and latitudes 31°29′N-32°27′N. The area and number of raingauge stations lying between different elevation ranges is given in Table 1.

The coefficient of variation of elevation field is found to be above 50%. It is also observed that the catchment is partly snow bound and during winter months about 70% of the area usually comes under the seasonal snow cover.

2. Correlation structure of precipitation field

Let there be n points of observations in a catchment of area A. Let us form the series of seasonal or annual precipitation for T years for each point and compute the product moment correlation between all possible (${}^nC_2$) pairs of series. If the values of r_i are plotted against the distance between the stations (s_i) the curve is expected to exhibit a decreasing trend ($dr/ds<0$). This type of curve is generally known as the correlation structure of seasonal/annual precipitation in the catchment.

Most of the authors have suggested the exponential structure to capture the above trend which is

$$r(s) = r_0\, e^{-bs} \tag{1}$$

TABLE 1

Area lying between different elevation range

Elevation range (m)	Area (A) (km²)	No. of raingauge stations	Coefficient of range of rainfall ($\frac{x_n - x_1}{x_n + x_1} \times 100$)
600	2507	1	—
600-1200	2455	7	51
1200-1800	3015	6	45
1800-2400	1275	5	6
2400-3000	1143	1	—
3000-3600	860	0	—
3600-4200	673	0	—
4200-4800	164	0	—
4800	419	0	—

Total area (A) = 12509 km², Mean height = 1725 m

whereas in some cases, where the rate of decrease is smaller, a modified Bessel function,

$$r(s) = sbk(sb) \quad (2)$$

has been observed as best fit. The distance 's' may be regarded as a random variable as it may assume different values between zero (close to it) and D (maximum possible distance between any two points in the basin) with frequency function $p(s)$. Form of $p(s)$ depends on the shape of the catchment. For instance, for a circular area (Erik Erikson 1972):

$$p(s) = 2s/R^2, \quad 0 \leqslant s \leqslant R \quad (3)$$

For other types of geometrical shape like square or rectangular, the expressions for $p(s)$ have been derived by some authors (Rodriguez *et al.* 1974). These are too complicated for practical computations. Moreover, the shape of an actual basin differs significantly from regular geometrical shape. As such, instead of going for exact sampling distribution, we propose to adopt Gamma p.d.f. for s. In that case, its frequency function may be written as :

$$p(s) = \frac{1}{\beta^{\gamma}\,\overline{|\gamma}}\, s^{\gamma-1}\, e^{-s/\beta}, \qquad 0 \leqslant s \leqslant \infty,\ \ \beta, \gamma > 0 \quad (4)$$

It may be seen that with $\bar{s} = 66$ km and $\sigma_s = 34$ km, the maximum density is concentrated in $0 \leqslant s \leqslant 200$ km range and $p(s > 200) \to 0$. It is on this logic, we see that a number of distributions, namely, normal, gamma, EV 1, 2 & 3 Pearson's etc which are valid for infinite ranges are also applied to rainfall and other variables which are bounded.

In this case, although a correction factor

$$c = \int_0^{\infty} p(s)\, ds \Big/ \int_0^{200} p(s)\, ds$$

may also be calculated, but for all practical purpose its value will be unity.

TABLE 2

Values of Ab^2 and $\bar{r}$ for various areas

Circle no.	Ab^2	$\bar{r}$
1	.011	.36
2	.027	32
3	.020	.36
4	.069	.28
5	.162	.21

The mean correlation over a given area 'A' is given by

$$E(r/A) = \bar{r} = \int r(s)\, p(s)\, ds$$

$$= \frac{r_0}{\beta^{\gamma}\,\overline{|\gamma}} \int_0^{\infty} s^{\gamma-1}\, e^{-s(b\beta+1)/\beta}\, ds$$

$$\bar{r} = \frac{r_0}{(1+b\beta)^{\gamma}} \quad (5)$$

b and $\bar{r}$ are constants for a particular area. The effect of area is included in expression (5) indirectly in the form of parameters r_0, b, β and γ as these parameters assume different values when 'A' changes. Thus, if we change the area, then b and $\bar{r}$ will also be changed. If these parameters are computed for different areas of similar shape, $\bar{r}$ may be expressed as a function of Ab^2, which is a dimensionless quantity. It is shown by Iturbe (1974), Ramanathan *et al.* (1981) etc that $\bar{r}$ is generally a decreasing function of Ab^2. To illustrate the variation of $\bar{r}$ with Ab^2 an exercise has been carried out using about 50 years annual rainfall data recorded by various stations in Himachal Pradesh. The entire region was subdivided into 5 circular zones. The form of p.d.f. for s as given by Eqn. (4) has been tried. $\bar{r}$, b and A were worked out for each circle. Table 2 gives $\bar{r}$ and Ab^2. The table indicates that $\bar{r}$ generally decreases with increasing Ab^2.

This relationship is useful in the sense that it ascertains an appropriate network density of a catchment based on its area (A) and the characteristic of correlation structure (b). This has been further elaborated in section 5.

3. Error in estimation of areal precipitation

(a) *For long-term areal precipitation*

In the studies by Iturbe (1974) and Ramanathan *et al.* (1981), it has been indicated that the variance of long-term areal precipitation can be split into 3 multiplicative components, namely,

(*i*) the variance of point rainfall process (σ^2),

(*ii*) temporal reduction factor $f(T)$, which depends on period of the data used for estimation of p_A, and

(*iii*) spatial reduction factor $\psi(n, \bar{r})$ which depends on the number of stations 'n' used for estimation of p_A. Thus,

$$V(p_A) = \sigma^2 f(T)\, \psi(n, \bar{r}) \quad (6)$$

TABLE 3

Station	Location			Annual rainfall (cm)	S.D. of rainfall (cm)	C.V. (%)
	Lat. (°N)	Long. (°E)	Height (m)			
Dharamsala	32° 16′	76° 23′	1211	336	184	54
Palampur	32° 08′	76° 32′	1217	256	96	38
Hamirpur	31° 42′	76° 32′	786	142	55	40
Kangra	32° 06′	76° 15′	701	196	53	27
Kulu	31° 57′	77° 07′	1236	100	30	30
Mandi	31° 43′	76° 56′	752	157	25	16
Jogindernagar	31° 55′	76° 45′	1221	223	56	25
Sundernagar	31° 32′	76° 53′	1193	159	27	17
Banjar	31° 38′	77° 20′	1522	109	28	26
Jubbal	31° 47′	77° 40′	1891	107	20	19

In a time series, with significant autocorrelation lag $1(\rho)$, $f(T)$ may be evaluated from :

$$f(T) = \frac{1}{T} \cdot \frac{1+\rho}{1-\rho} \tag{7}$$

$\psi(n, \bar{r})$ is a function of n and $\bar{r}$; it may be shown that

$$\psi(n, \bar{r}) = \frac{1+(n-1)\,\bar{r}}{n}, \qquad \textit{vide} \text{ Appendix (B)}$$

$$= \frac{1-\bar{r}}{n} + \bar{r} \tag{8}$$

(b) *For a rainfall event*

If p_i $(i=1, 2, 3, \ldots\ldots\ldots, n)$ are the rainfall amounts recorded at n points during a rainfall event and the areal mean is expressed by

$$p_A = \frac{1}{n} \sum_{i=1}^{n} p_i \tag{9}$$

the variance of p_A is subjected to only spatial reduction factor. Therefore,

$$V(p_A) = \sigma^2\, \psi(n, \bar{r})$$

$$= \sigma^2 \left(\frac{1-\bar{r}}{n}\right) \tag{10}$$

4. Error in interpolation

There are several methods suggested for spatial interpolation of precipitation, *e.g.*, a triangular grid network in which the stations are located at apexes. It has been shown by Kagan (1972) that in such cases the error of interpolation depends on :

(*i*) the variability of point rainfall process,

(*ii*) the network density 'n',

(*iii*) the spacing between two stations or areas of the catchment, and correlation structure of precipitation field.

It may be estimated from the following expression:

$$Z_{\text{int}} = C_v \sqrt{\frac{1-r_0}{3} + 0.526\, r_0\, \sqrt{A/n}} \tag{11}$$

TABLE 4

Bivariate frequency distribution of r and s

s	r: −.2 — −0	0 — .2	.2 — .4	.4 — .6	.6 — .8	.8 — 1.0	f_s
0-20				1	1		2
20 - 40			1	2	3	1	7
40 - 60		6	3	4	3		16
60 - 80		1	1	2	1		5
80 - 100	1	2	3	1	1		8
100-120			2	1			3
120-140		1	1				2
140-160		1	1				2
f_r	1	11	12	11	9	1	45

where $C_v = \sigma_p / p_A$, r_0 and b are the parameters of correlation structure and n is the number of raingauge stations in an area A. It is mentioned that for a square area A the spacing between two stations may be $l = \sqrt{A/n}$ whereas for a triangular area A,

$$l = 1.07\, \sqrt{A/n}$$

5. A case study : Beas catchment

In the present study, about 18 years' precipitation data (1952-1969) have been utilised for the ten selected stations given in Table 3.

The bivariate frequency distribution, $f(r, s)$ of different correlations (r) and distances (s) as computed from these ten precipitation series is shown in Table 4.

The mean correlations for various distances are :

s (km) : 10, 30, 50, 70, 90, 110, 130, 150

r : 0.75, 0.61, 0.34, 0.42, 0.28, 0.37, 0.20, 0.20

The exponential form fitted to the above data of s and r is

$$r(s) = r_0\, e^{-bs} \tag{12}$$

The estimates of 'b' and r_0 have been worked out and are given as

$$r_0 = 0.84 \text{ and } b = 0.0098 \text{ km}^{-1}.$$

Therefore,

$$r(s) = .84\, e^{-.0098\, s} \tag{13}$$

Assuming a two parameter gamma distribution for the random variable s, the frequency function is given by Eqn. (4). Using the method of moments for the estimation of parameters β and γ, we get

$$\gamma = 4/g^2 \text{ and } \beta = \bar{s}/\gamma \tag{14}$$

where g is the coefficient of skewness $= \mu_3/\mu_2^{3/2}$

From the stations selected in the catchment

$\bar{s} = 66.4$ km, $\sigma_s = 34.1$ km.

TABLE 5

n	T
1	17
2	12
3	11
5	9
10	8
100	8

Therefore,

$\gamma=8.0$ and $\beta=8.3$.

The mean correlation of precipitation field $\bar{r}$ may be computed from Eqn. (6) which comes out to be $\bar{r}=0.45$.

(*i*) *The computation of temporal and spatial reduction factors*

The following catchment characteristics related to the point precipitation process have been estimated using 18 years data ($T=18$) for 10 stations ($n=10$) for Beas catchment. The mean precipitation $\bar{p}=172.2$ cm; standard deviation $\sigma_p=79.0$ cm and C.V. $=46\%$.

The combined reduction factor $f(T)$. $\psi(n, \bar{r})$ to the variance of p_A is provided in Table 7 which has been worked out by assuming $\rho=.25$ and $\bar{r}=.45$.

For reducing $V(p_A)$ to 10 % of σ_p^2 we have the following trade off between n and T which is presented in Table 5 and is given by $T=7.51+9.18/n$. Thus, for achieving 90% of accuracy in estimation of long-term areal precipitation, the period of data is more important than the density of network.

(*ii*) *Estimation of areal mean of an individual storm*

It has been derived in appendix that the expression for $V(p_A)$ when p_A is being estimated using n point values of rainfall over an area 'A' is given by

$$V(p_A)=\frac{\sigma_p^2}{n}(1-\bar{r})$$

For Beas catchment $\bar{r}=0.45$ and $V(p_A)=0.55\,\sigma_p^2/n$. Taking relative error of estimate as 0.1, we have

$$\frac{\sqrt{V(p_A)}}{p_A}=\sqrt{\frac{0.55}{n}}\cdot\frac{\sigma_p}{p_A}=0.1$$

$$\therefore\ 0.46\sqrt{\frac{0.55}{n}}=0.1$$

or $n=12$

It may be inferred from the above that 12 precipitation gauges would be required to provide the estimate of areal precipitation with 10% error. It may also be noted that if the correlation structure of precipitation field is not taken into account, the value of n would be higher for the same level of accuracy.

TABLE 6

Depth-area relationship

Area (km²)	$p_A/p=\sqrt{\bar{r}}$
2000	0.86
4000	0.74
6000	0.68
8000	0.63
10000	0.60
12000	0.59

(*iii*) *Areal reduction of precipitation*

If p is the point rainfall and p_A is the areal rainfall over an area A around that point, then Rodriguez *et al.* (1974) have shown the relationship between the two is given by

$$p_A=\sqrt{\bar{r}}\cdot p \tag{15}$$

where $\bar{r}$ is the expected value of correlation between two randomly chosen points in area A.

For Beas catchment, the Eqn. (15) has been used to derive the depth area relationship. The results are provided in Table 6 and in Fig. 2.

(*iv*) *Error in interpolation*

For a triangular grid with spacing $l=1.07\sqrt{A/n}$ the relative error of spatial interpolation in Beas catchment may be computed from Eqn. (11).

$$Z_{int}=0.45\sqrt{0.53+\frac{0.4789}{\sqrt{n}}}$$

n	Z_{int}
1	.335
2	.288
5	.238
10	.208
100	.146

6. Summary

(*i*) The parameters $r_0=0.84$ and $b=1/102$ km of the correlation structure of the precipitation field in Beas catchment suggest that r converges at short distances. This may be attributed to the higher variability as observed in a mountainous catchment (C.V.$=46\%$).

(*ii*) A lower number of precipitation gauges are required for estimation of areal precipitation with same degree of accuracy (e) if $\bar{r}$ is taken into account. It follows from the following :

(*a*) if r is not considered

$$n=\left(\frac{\text{C.V.}}{e}\right)^2 \tag{16}$$

TABLE 7

T	n: 1	2	3	4	5	10	15	20	25	30	50	100
2	.835	.605	.528	.490	.468	.422	.406	.398	.394	.391	.385	.380
5	.334	.242	.211	.196	.187	.169	.162	.159	.158	.156	.154	.152
10	.167	.121	.106	.098	.094	.084	.081	.080	.079	.078	.077	.076
15	.111	.080	.070	.065	.062	.056	.054	.053	.052	.052	.051	.051
20	.084	.061	.053	.049	.047	.042	.040	.040	.039	.039	.038	.038
25	.067	.049	.042	.039	.037	.034	.033	.032	.032	.031	.031	.030
30	.056	.040	.035	.033	.031	.028	.026	.026	.026	.026	.026	.026
35	.048	.035	.030	.028	.027	.024	.023	.023	.023	.022	.022	.022
40	.042	.030	.026	.024	.023	.021	.019	.020	.020	.020	.019	.019
45	.037	.027	.023	.022	.021	.019	.018	.018	.017	.018	.017	.017
50	.033	.024	.021	.020	.019	.017	.016	.016	.016	.016	.015	.015

Reduction factor $f(T)\,\psi(n, r) = \left\{\frac{1}{T}\cdot\frac{1+\rho}{1-\rho}\right\}\left\{\frac{1+(n-1)\bar{r}}{n}\right\}$; $(\rho=0.25\ ;\ \bar{r} = 0.45)$

(*b*) with the consideration of $\bar{r}$,

$$n = \left(\frac{\text{C.V.}}{e}\right)^2 (1-\bar{r}),$$

where e is the acceptable error (%).

(*iii*) It is also worth noticing that while estimating areal mean (p_A) for a rainfall event, the spatial reduction factor for $V(p_A)$ cannot be less than $\bar{r}$. Thus, there has to be an optimum value of n, which minimises the product of spatial and temporal reduction factors. For $V(p_A) = 0.1\sigma_p{}^2$ the trade off between n and T is given by $T=7.5+9.2/n$.

(*iv*) The relative error of spatial interpolation for a triangular grid in respect of Beas catchment may be reduced from 33% to 15% by raising the number of gauges from 1 to 100.

(*v*) Taking the areal reduction factor for point to areal rainfall as $p_A/p = \sqrt{\bar{r}}$, the areal rainfall in Beas catchment appears to be 78% of point rainfall for one quarter of the area, 70% for half of the area, 65% for 3/4th of the area and 59% for the entire catchment.

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APPENDIX

Error in estmation of areal precipitation

(*A*) *Rainfall event*

Let p_i (i=1, 2, ..., n) be the precipitation at point i. The estimate of mean areal precipitation.

$$p_A = \frac{1}{A}\int_A p(x_i)\,dx_i \tag{1}$$

is given approximately by

$$\hat{p}_A = \frac{1}{n}\sum_{i=1}^{n} p_i \tag{2}$$

The variance of the areal precipitation is

$$\begin{aligned} V(p_A) &= E(\hat{p}_A - p_A)^2 \\ &= E\left\{\frac{1}{n^2}\left(\sum p_i\right)^2\right\} - \\ &- E\left\{\frac{2}{nA}\left(\sum p_i\right)\int_A p(x_i)\,dx_i\right\} + \\ &+ E\left\{\frac{1}{A}\int_A p(x_i)\,dx_i\right\}^2 \end{aligned} \tag{3}$$

The expression on R. H. S. has 3 terms, *viz.*,

$$\mathrm{I} = \frac{1}{n^2} E \left(p_1 + p_2 + \ldots + p_n \right)^2$$

$$= \frac{1}{n^2} E \left\{ \sum_{i=1}^{n} p_i^2 + \sum_{i \neq j}^{n} \sum p_i p_j \right\}$$

Assuming without loss of generality the deviation from respective means we get,

$$\mathrm{I} = \frac{1}{n^2} \left[n\sigma^2 + \sigma^2 \sum_{i \neq j}^{n} \sum r_{ij} \right]$$

$$= \frac{\sigma^2}{n} \left[1 + (n-1)\bar{r} \right] \tag{4}$$

where σ^2 and $\bar{r}$ represent variance and mean correlation for the catchment respectively,

$$\mathrm{II} = \frac{-2}{nA} E \left[\left(\sum_{i=1}^{n} p_i \right) \int_A p(x_i)\, dx_i \right]$$

$$= \frac{-2}{nA} \cdot n\sigma^2 \bar{r} \cdot A = -2\sigma^2 \bar{r} \tag{5}$$

$$\mathrm{III} = \frac{1}{A^2} E \left\{ \int_A p(x_i)\, dx_i \right\}^2$$

$$= \frac{\sigma^2}{A^2} \int_A \int_A r_{ij}\, dx_i\, dx_i = \frac{\sigma^2}{A^2} \cdot \bar{r} A^2 = \sigma^2 \bar{r} \tag{6}$$

Adding (4), (5) and (6), we get

$$V(p_A) = \frac{\sigma^2}{n} (1 - \bar{r}) \tag{7}$$

(B) Long-term areal precipitation

If rainfall process in the catchment is represented by $p(s, t)$, where 's' & 't' are the coordinates of space and time, the estimate of long term areal precipitation :

$$p_A = \underset{T \to \infty}{\mathrm{Lt}} \frac{1}{AT} \sum_{t=1}^{T} \int_A p(s_i, t)\, ds_i \tag{8}$$

is given by

$$\hat{p}_A = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} p(s_i, t) \tag{9}$$

$$\therefore V(p_A) = E \left[\frac{1}{nT} \sum_i \sum_t p(s_i, t) - \underset{T \to \infty}{\mathrm{Lt}} \frac{1}{AT} \sum_t \int_A p(s_i, t)\, ds_i \right]^2$$

$$= E \left[\frac{1}{nT} \sum_i \sum_t p(s_i, t) \right]^2 - 2E \left[\left\{ \underset{T \to \infty}{\mathrm{Lt}} \frac{1}{AT} \sum_t \int_A p(s_i, t)\, ds_i \right\} \times \left\{ \frac{1}{nT} \sum_i \sum_t p(s_i, t) \right\} \right] + \underset{T \to \infty}{\mathrm{Lt}} E \left[\frac{1}{AT} \sum_t \int_A p(s_i, t)\, ds_i \right]^2 \tag{10}$$

The three expressions of R. H. S. may be further simplified as shown below :

$$\mathrm{I} = \frac{1}{n^2 T^2} E \left[\sum_t \sum_i p^2(s_i, t) + \sum_t \sum_{i \neq j} p(s_i, t)\, p(s_j, t) + \sum_{t \neq t'} \left\{ \sum_i p(s_i, t)\, p(s_i, t') + \sum_{i \neq j} p(s_i, t)\, p(s_j, t') \right\} \right]$$

$$= \frac{1}{n^2 T^2} \left[nT\sigma^2 + Tn(n-1)\bar{r}\sigma^2 + \sum_{t \neq t'} \left\{ n\sigma^2 \rho^{|t-t'|} + n(n-1) \rho^{|t-t'|} \bar{r}\sigma^2 \right\} \right] \tag{11}$$

where ρ is 1st autocorrelation coefficient.

Assuming $\rho_L = \rho^L$; $L = 1, 2, \ldots, T$,

$$\mathrm{I} = \frac{n\sigma^2}{n^2 T^2} \left[T \left\{ 1 + (n-1)\bar{r} \right\} + \left\{ 1 + (n-1)\bar{r} \right\} \sum_{t \neq t'} \rho^{|t-t'|} \right]$$

$$\text{Now } \sum_{t \neq t' = 1}^{T} \rho^{|t-t'|} = 2 \left[(\rho + \rho^2 + \ldots + \rho^{T-1}) + (\rho + \rho^2 + \ldots + \rho^{T-2}) + \ldots (\rho + \rho^2) + \rho \right]$$

$$= \frac{2\rho}{1-\rho} \left[(T-1) - \frac{\rho}{1-\rho} (1 - \rho^{T-1}) \right] \tag{12}$$

$$\therefore V(p_A) = \sigma^2 \psi(n, \bar{r}) \cdot f(T) \tag{13}$$

where $\psi(n, \bar{r}) = \dfrac{1 + (n-1)\bar{r}}{n}$ and

$$f(T) = \frac{1}{T} + \frac{2\rho}{(1-\rho)T^2} \left[(T-1) - \frac{\rho}{1-\rho} (1 - \rho^{T-1}) \right]$$

$$\simeq \frac{1}{T} \cdot \frac{1+\rho}{1-\rho} \text{ for large } T \text{ and neglecting terms in } T^2$$