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# A deterministic non-linear model for rainfall-runoff relation

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ABSTRACT. A non-linear model for the rainfall run-off relation is discussed by using the Laplace's transform in the equation of continuity and storage. The equation of continuity has been converted into a $n$th order algebraic equation. A simple integration method has also been indicated for the constant inpr function. Some particular cases are also discussed as a verification of the result, and lastly the linear moden has been deduced as a special case from the non-linear model. The results are in close agreement with that of observed hydrograph.

## 1. Introduction

Time invariant and time variant linear models representing the rainfall-runoff relationship have been presented and discussed by many authors in recent time. Authors, like Nash (1957), Kulandaiswamy (1964), Singh (1964), Chiu and Bittler (1969), Prasad (1967), Mathur (1972), and Balek and Jokl (1974) have considered either the first order or the second order linear differential equation for their models. Guha (*see* Ref.) has shown that the time dependent model in which rainfall-runoff relation is related by a second order linear differential equation can be solved analytically and the output can be related by an integral equation, the solution of which has been achieved by fractional integration technique. We have also shown that result of Balek and Jokl (1974) is a special case wherein the coefficients in the rainfall-runoff relation, namely,

$$A(t)\frac{d^2Q}{dt^2} + B(t)\frac{dQ}{dt} + C(t)\,Q(t) = I(t)$$

are treated as constants.

In this discussion, the non-linear model has been solved by reducing the differential equation connecting the output and input into a $n$th order ordinary algebraic equation.

The variation of the index parameter $n$ in $S = KQ^n$ has been discussed by different authors like Handerson (1966), Henderson and Wooding (1964) among the others. They have shown that it varies from $\frac{1}{2}$ to 3 for different types of flow. In our discussion, we put only the restriction that $n \neq 0$. We have also deduced and verified that the solution of the linear model can be obtained from the non-linear model discussed here by simply substituting $n=1$, Ding (1967) has discussed the non-linear model in the case of constant input only.

## 2. Formulation of the problem

The equation of continuity is given by :

$$I(t) - Q(t) = \frac{dS}{dt} \tag{1}$$

where, $I$ is the effective rainfall rate, $Q$ is the discharge rate and $S$ is the storage. Let us assume the relation :

$$S = KQ^n \tag{2}$$

The dimension of $K$ here is given by :

$$L^{(3-3n)}.\quad T^n \tag{3}$$

$n$ usually varies from $\frac{1}{2}$ to 3 right from the orifice to the laminar flow. Using Eqns. (1) and (2) we obtain,

$$Q^{n-1}\frac{dQ}{dt} + \frac{1}{Kn}\,Q = \frac{1}{Kn}\,I \tag{4}$$

assuming $n \neq 0$

TABLE 1

Bridge No. 15 Region 3 F

| Period (hr) | Gross rainfall (cm) | Effective rainfall (cm) | Observed hydrograph (cusecs) | Computed hydrograph (cusecs) | |
|---|---|---|---|---|---|
| | | | | $n=1$ | $n=2$ |
| (1) | (2) | (3) | (4) | (5) | |
| 1 | 0·40 | 0·0 | 1 | ·8 | 11·2 |
| 2 | 0·10 | 0·088 | 10 | 9·2 | 11·2 |
| 3 | 0·30 | 0·288 | 27 | 26·9 | 31·3 |
| 4 | 0·10 | 0·088 | 53 | 53·1 | 56·8 |
| 5 | 0·00 | 0·00 | 70 | 61·7 | 70·4 |
| 6 | 0·00 | 0 | 87·9 | 73·5 | 93·1 |
| 7 | | | 83·0 | 65·1 | 74·7 |
| 8 | | | 60·0 | 58·7 | 68·4 |
| 9 | | | 46·0 | 39·1 | 52·8 |
| 10 | | | 129·0 | 29·6 | 44·0 |
| 11 | | | 21·0 | 23·5 | 35·24 |
| 12 | | | 18.0 | 13·8 | 26·4 |
| 13 | | | 15·0 | 11·3 | 17·6 |
| 14 | | | 13·0 | 10·8 | 15·8 |
| 15 | | | 11·0 | 8·3 | 13·6 |
| 16 | | | 9.0 | 5.6 | 10.2 |
| 17 | | | 7·0 | 4·0 | 8·8 |
| 18 | | | 5·0 | 3·6 | 6·3 |

Taking the Laplace transform in (4) we get,

$$p\bar{Q^n}(p) + \frac{1}{n}\ \bar{Q}(p) = \frac{1}{K}\ \bar{I}(p) \qquad (5)$$

where,

$$\bar{Q}(p) = \int_0^\infty Q(t)\ e^{-pt}\ dt \qquad (6)$$

$$\bar{I}(p) = \int_0^\infty I(t)\ e^{-pt}\ dt \qquad (7)$$

$$\bar{Q}^n(p) = \int_0^\infty Q^n(t)\ e^{-pt}\ dt \qquad (8)$$

The Eqn. (5) is a $n$th order algebraic equation and consequently can be solved exactly. It can easily be shown that when $n=1$, *i.e.*, in the case of linear model; when $I$ is constant, the solution reduces to $Q(t) = I(1-e^{-t/K})$ which is in agreement with the well known result.

Similarly for $n = 1$ and $I = I(t)$ we can easily obtain from the Eqn. (5),

$$Q(t) = \frac{1}{K}\int_0^t I(t)\ e^{-t/K}\ dt \qquad (9)$$

which is also in agreement with the well known result. The response function in each of the above cases is :

$$u(t) = \frac{1}{K}\ e^{-t/K} \qquad (9a)$$

For $n = 2$, *i.e.*, for the non-linear model the solution is given by Eqn. (5)

$$Q/I + \log(I - Q/I) = -t/2\,KI \qquad (10)$$

which can also be verified from the equation of continuity. It is to be noted that $Q$ and $I$ are related implicitly and non-linearly in the case.

For $n = 3$, $I = I$(constant), the solution is given by

$$Q/I + \frac{1}{2}(Q/I)^2 + \log(1 - Q/I) = -\frac{t}{3I^2K} \qquad (11)$$

In general for $n = m$, where $m \neq 0$, the corresponding solution will be :

$$\frac{1}{m}(Q/I)^m + \log(1 - Q/I) = -\frac{t}{mKI^{m-1}} \qquad (12)$$

Exactly in the same manner, we can consider the most generalised non-linear model, *viz.*,

$$S = a_1Q(t) + a_2Q^2(t) + \cdot\ \cdot\ \cdot + a_n Q^n(t) \qquad (13)$$

or
$$S = \sum_{i=1}^{n} a_i \, Q^i(t) \tag{14}$$

In this case, also, we encounter with a non-linear first order ordinary differential equation of similar type and consequently can be solved exactly in the like-manner, *via*, the Laplace's transform. It is to be noted that the index parameter $n$ satisfies the inequality :

$$\tfrac{1}{2} \leqslant n \leqslant 3$$

for the actual water-sheds. In Table 1, we have computed the discharges according to the model for $n = 1$ and $n = 2$ only and one can take the fractional values for $n$ also.

Again, we can write equation of continuity as

$$\frac{d\,(Q^n)}{I - Q} = \frac{1}{K}\,dt \tag{15}$$

Integrating, we obatain

$$G\,(u, n) - G\,(u_0, n) = \frac{t}{K}\, I^{1-n} \tag{16}$$

where,
$$G\,(u, n) = \int_{u_0}^{u} \frac{d\,u}{1 - u^{1/n}} \tag{17}$$

Let, the initial condition be such that,

$$G\,(u_0, n) = 0 \tag{18}$$

So that, we get from Eqn. (17)

$$G\,(u, n) = \frac{t}{K} I^{1/n} \tag{19}$$

For $n = 1$, we get,

$$Q\,(t) = I\,\left(1 - e^{-t/K}\right) \tag{20}$$

which is the same result as obtained earlier.

## 3. Computations

The above model results have been applied for both $n=1$ and $n=2$ over the 3 F region at bridge No. 15 (Table 1). The column (2) indicates the total rainfall amount at an interval of 1 hr obtained by SRG while the column (3) is the excess rainfall. Columns (4) and (5) are the observed and reproduced hydrographs.

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### Appendix

**Derivations of the equations (10), (11) and (12)**

For $n = 2$, we get, from Eqn. (4)

$$\int_0^Q \frac{d\,(Q^2)}{I-Q} = \int_0^t \frac{dt}{K} \tag{21}$$

Let us take,

$$Q/I = u$$

*i.e.,* $$dQ = I\,du$$

From Eqn. (21) we get,

$$\frac{Q}{I} + \log\,(1-Q/I) = -t/2\,K\,I \tag{22}$$

For $n = 3$, we have in the same manner,

$$Q/I + \frac{(Q/I)^2}{2} + \log\,(1-Q/I) = -\frac{t}{3I^2K} \tag{23}$$

In general, for $n = m$, we obtain

$$\sum_{m=2}^{m-1} \frac{(Q/I)^{m-1}}{m-1} + \log\,(1-Q/I) = -\frac{t}{mKI^{m-1}} \tag{24}$$

$$m \neq 1, \text{ and } \neq 0.$$