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On the dispersion of love waves in a continuously stratified layered earth

S. N. CHATTERJEE

Meteorological Office, New Delhi

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ABSTRACT. Love wave dispersion equation in a continuously heterogeneous layered-flat-earth has been studied. Different types of plausible variation of elastic parameters in each of the layers have been considered and the corresponding displacements have been obtained. Thomson Harkell Matrix method has been employed to derive the dispersion equation of *SH* surface waves in such a layered earth.

1. Introduction

Investigation of the earth's interior by the method of dispersion of surface waves has special significance, particularly in regions where other geophysical methods cannot be used. Study of surface waves is used in the determination of structure of a medium along which they propagate, in the determination of the parameters of the source, in the identification of sub-terranean explosions and in tracking the storms in sea with the help of microseisms.

Since continuous variations of elastic parameters and density are known to exist in the crust and mantle, various authors, *viz.*, Meissner (1921), Jeffreys (1928), Bateman (1928), Matuzawa (1929), Satô (1952), Mal (1962), Maulick (1965), Avtar (1967), Sinha (1969), Chatterjee (1969, 1971), Bhattacharya (1970, 1972) considered the propagation of love waves in a medium in which elastic parameters and density are functions of depth.

Kailis-Borok, Neigaus and Shkadinskaya (1965), and Vlaar (1966) considered love wave propagation in an elastic and isotropic half space in which the elastic parameters and density are piecewise continuous functions of depth.

Thomson (1950) introduced a matrix method to determine the transmission and reflection coefficients of plane body waves through a stratified solid medium. Haskell (1953) applied Thomson's matrix formulation to the problems of surface waves and developed a convenient method to compute dispersion for a multilayered medium composed of any number of plane paralleled layers. This method, known as Thomson-Haskell method has been used by Dorman, Ewing and Oliver (1960) to calculate surface wave dispersion for a number of continental and oceanic crust mantle structures. Press, Harkrider and Seafeldt (1961), using this method with more advanced computer, greatly improved the speed of computation. In recent years the speed of computation has further been improved successively by Thrower (1965), Randak (1967), Watson (1970), Biswas and Knopoff (1970).

The Thomson-Haskell matrix method, has been generalized for transversely isotropic media by Anderson (1961), which was further extended by Saastamoinem (1969) to a multilayered semi-infinite medium where in each layer the modulus of rigidity and density are functions of depth.

Since regional variation of structures are well known and the structure of the upper mantle is known to be complex, possibility of various types of variation in elastic parameters exists. In the present paper the author investigated the dispersion of love wave in a multilayered inhomogeneous half space in which each layer is piecewise continuous and the variations of elastic parameters follow different types of mathematical laws. Thomson-Haskell's technique has been employed in the determination of frequency equation.

2. Determination of frequency equation

Let us consider a layered flat earth, with welded contact, each layer being inhomogeneous. Let the laws governing the variation of rigidity and shear wave velocity are as follows :

Case I

$$\mu = \mu_0 \exp (p\, z^2) \quad \beta = \beta_0 / (1 \pm q^2 z^2)^{1/2} \qquad (1)$$

Case II

$$\mu = \mu_0 \exp (2\, pz) \quad \beta = \beta_0 (1 - qz)^{1/2} \qquad (2)$$

Case III

$$\mu = \mu_0 (1 + p z)^l \quad \beta = \beta_0 (1 + p z)/[(1 + p z)^2 - \alpha (1 + p z) - q]^{1/2} \qquad (3)$$

Case IV

$$\mu = \mu_0 (1 + p z)^2 \quad \beta = \beta_0 / (1 - q \cos 2 z)^{1/2} \qquad (4)$$

where, μ_0, β_0, l, p, α and q are all constants.

The z-axis is taken vertically downwards. The geometry of the problem and direction of axes are presented in Fig. 1. For love wave $u=\omega=0$, $v=v(x, z)$ and in the absence of body forces, the equation of motion for SH type surface waves is a vertically inhomogeneous layer (Ewing, Jardetzky and Press 1957) in

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}$$

or

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z} \left(\mu \frac{\partial v}{\partial z}\right) \qquad (5)$$

The symbols have their usual significance. Since μ is a function of z only the equation (5) transforms into

$$\rho \frac{\partial^2 v}{\partial t^2} = \mu \nabla^2 v + \frac{d \mu}{dz} \frac{d v}{d z} \qquad (6)$$

If we assume

$$V = \sqrt{\mu} . v$$

the equation (6) reduces to

$$\rho \frac{\partial^2 V}{\partial t^2} = \mu \nabla^2 V + \left[\frac{1}{4 \mu^2} \left(\frac{d \mu}{d z}\right)^2 - \frac{1}{2} \frac{d^2 \mu}{d z^2}\right] V \qquad (7)$$

If we assume the motion to be S.H.M., we can put

$$V = \phi (z) \exp \{ik (x - ct)\} \qquad (8)$$

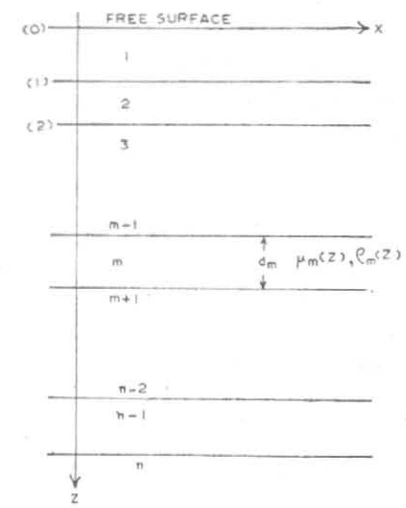

Fig. 1. Direction of axes and geometry of the problem

then $\phi (z)$ will be the solution of the differential equation

$$\frac{d^2 \phi}{d z^2} + \left[\frac{\omega^2}{\beta^2} - k^2 + \frac{1}{4\mu^2} \left(\frac{d\mu}{dz}\right)^2 - \frac{1}{2 \mu} \frac{d^2 \mu}{d z^2}\right] \phi = 0 \qquad (9)$$

Case I

Substituting the values of μ and β in (7) we get

$$\frac{d^2 \phi}{dz^2} + \left[\left(\pm \frac{\omega^2}{\beta_0^2} q^2 - p^2\right) z^2 + \left(\frac{\omega}{\beta_0^2} - k^2 - p\right)\right] \phi = 0 \qquad (10)$$

Taking the lower sign, equation (10) reduces to

$$\frac{d^2 \phi}{d z^2} + \left[- a_1^2 - b_1^2 z^2 \right] \phi = 0 \qquad (11)$$

where,

$$a_1^2 = - \left(\frac{\omega^2}{\beta_0^2} q^2 + p^2 \right) \text{ and}$$

$$b_1^2 = \left(\frac{\omega^2}{\beta_0^2} - k^2 - p\right) \qquad (12)$$

If we assume b_1 to be imaginary, then $(\omega^2/\beta_0^2) - k^2 - p$ is negative.

Again since $\omega = kc$, this condition reduces to

$$k^2 \left(\frac{c^2}{\beta_0^2} - 1 \right) < p \qquad (13)$$

The following substitution

$$\xi = \sqrt{2a_2}\, z \qquad b_1 = ia_2, \quad i = \sqrt{-1} \qquad (14)$$

reduces equation (11) to

$$\frac{d^2 \phi}{d \xi^2} + \left[\frac{1}{4} \xi^2 - l \right] \phi = 0 \qquad (15)$$

where,

$$l = a_1^2 / 2a_2 = (\omega^2 / \beta_0^2 + p^2) / 2 \sqrt{(k^2 + p - \omega^2 / \beta_0^2)} \qquad (16)$$

Equation (15) is satisfied by the parabolic cylindrical functions and the solution is given by (Abramobitz and Stegun 1965)

$$\phi = A_3\, W(l, \xi) + B_3\, W(l, -\xi) \qquad (17)$$

Now,

$$W(l \pm \xi) = \frac{(\cosh \pi l)^{1/4}}{2\sqrt{\pi}} \{G_1 \Psi_1 \mp \pm\sqrt{2}\, G_2 \Psi_2\} \qquad (18)$$

where,

$$G_1 = \left| \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2} i l\right) \right|$$

$$G_2 = \left| \Gamma\left(\tfrac{3}{4} + \tfrac{1}{2} i l\right) \right|$$

$$\Psi_1 = 1 + \frac{l\xi^2}{2!} + \left(l^2 - \frac{1}{2}\right)\frac{\xi^4}{4!} + \left(l^3 - \frac{7}{2}\right)\frac{\xi^6}{6!} + \left(l^4 - 11l^2 + \frac{15}{4}\right)\frac{\xi^2}{8!} + \ldots \qquad (19)$$

$$\Psi_2 = \xi + \frac{l\xi^3}{3!} + \left(l^2 - \frac{2}{3}\right)\frac{\xi^5}{5!} + \left(l^3 - \frac{13}{2} l\right)\frac{\xi^7}{7!} + \left(l^4 - 17l^2 + \frac{63}{4}\right)\frac{\xi^9}{9!} + \ldots$$

and nonzero coefficients of c_n, of $\xi^n/n!$ are connected by

$$c_{n+2} = lc_n - \tfrac{1}{4}n(n-1)c_{n-2} \qquad (20)$$

As the wronskian of $W(l, \xi)$ and $W(l, -\xi)$ is nonzero, they form a fundamental system of solutions, when $\xi \to \infty$ both $W(l, \xi)$ and $W(l, -\xi)$ oscillate with decreasing amplitudes which slowly tend towards zero as $1/\sqrt{\xi}$ where, $a_2 \to 0$, so that $l \to \infty$ in the range of intrerest $k^2 + p > \omega^2/\beta_0^2$, we have

$$W(l, \xi) \simeq \frac{1}{2^{\frac{3}{4}}} \sqrt{\left(\frac{G_1}{G_2}\right)} \exp\left\{ -(k^2 + p - \omega^2/\beta_0^2) \right\},$$

$$W(l, -\xi) \simeq \frac{1}{2^{\frac{3}{4}}} \sqrt{\left(\frac{G_1}{G_2}\right)} \exp\left\{ k^2 + p - \omega^2/\beta_0^2 \right\} \qquad (21)$$

Thus we approach the case of homogeneous medium provided p is neglected. The asymptotic behaviour of $W(l, \pm \xi)$ shows in order to satisfy Sommerfeld's radiation condition.

$$\phi = A_3\, W(l, \xi) \qquad (22)$$

This is true provided the half space is also governed by the distribution given in case I.

When we take into consideration the upper sign of the equation (10), we substitute,

$$\xi = (\omega^2 q^2/\beta_0^2 + p^2)^{\frac{1}{3}} z^2,$$

$$\Psi = z^{\frac{1}{2}} \phi \qquad \text{and}$$

$$l = -\tfrac{1}{4}(\omega^2 q^2/\beta_0^2 + p^2)^{-1} (k^2 + p - \omega^2/\beta_0^2) \qquad (23)$$

in the equation (10) which then transforms into

$$\frac{d^2 \Psi}{d \xi^2} + \left[-\frac{1}{4} + \frac{l}{\xi} - \frac{(\frac{1}{4})^2 - \frac{1}{4}}{\xi^2} \right] \Psi = 0 \qquad (24)$$

This is a Whihaker's equation. Remembering that for $z \to \infty$ the displacement must vanish, we have the solution

$$\phi = A\, \xi^{-\frac{1}{4}}\, W_{l, \frac{1}{4}}(\xi) \qquad (25)$$

Case II

In this case substituting the values of μ and β in the equation (9), we get

$$\frac{d^2 \phi}{d z^2} + \left[\frac{\omega^2}{\beta_0^2}(1 - qz) - (p^2 + k^2) \right] \phi = 0 \qquad (26)$$

with the change of variable

$$\xi = \left(\frac{\omega^2 q}{\beta_0^2}\right)^{-\frac{2}{3}} \left[-\frac{\omega^2}{\beta_0^2}(1 - qz) + (p^2 + k^2) \right] \qquad (27)$$

equation (26) reduces to

$$\frac{d^2\phi}{d\xi^2} - \xi \phi = 0 \qquad (28)$$

The general solution of (28) is given by

$$\phi = A_3 A_i(\xi) + B_3 B_i(\xi) \tag{29}$$

where $A_i(\xi)$ and $B_i(\xi)$ are Airy functions of the first and second kind respectively.

Case III

Substituting the value of μ and β in the differential equation (9) and changing the variable

$$\xi = \frac{(1+pz)}{p}\left(\frac{\omega^2}{\beta_0^2} - k^2\right)^{\frac{1}{2}} \tag{30}$$

equation (9) transforms into

$$\frac{d^2\phi}{d\xi^2} + \left[1 - \frac{2\eta}{\xi} - \frac{L(L+1)}{\xi^2}\right]\phi = 0 \tag{31}$$

where,

$$\eta = \frac{\omega^2 \alpha}{2p}\left(\frac{\omega^2}{\beta_0^2} - k^2\right)^{-\frac{1}{2}} \text{ and}$$

$$L(L+1) = \frac{1}{4\beta_0^2 p^2}\left\{l(l-2)\, p^2 \beta_0^2 + 4\omega^2 q\right\} \tag{32}$$

assuming L to be a non negative integer.

The equation (31) is a Coulomb wave equation which has a regular singularity at $\xi=0$, *i.e.*, for $z=-1/p$ which includes $L(L+1)$ and $-L$. The general solution is given by

$$\phi = A\, F_L(\eta, \xi) + B\, G_L(\eta, \xi) \tag{33}$$

Case IV

Equation (9), on substitution transforms into

$$\frac{d^2\phi}{dz^2}(p - 2q\cos 2z)\,\phi = 0 \tag{34}$$

where,

$$p = \frac{\omega^2}{\beta_0^2} - k^2 + \frac{p^2}{4} \text{ and}$$

$$q = \omega^2 p/\beta_0^2 \tag{35}$$

Equation (34) is the Mather's equation and has the periodic solution

$$\phi = \sum_{m=0}^{\infty}(A_m \cos mz + B_m \sin mz) \tag{36}$$

Remembering that the displacement

$$v = \mu^{-1/2}\,\phi(z)\exp\{ik(x-ct)\}$$

we are going to apply Thomson-Haskell Matrix method for the layered structure of the earth. There are n differential equations, as obtained for each layer for different cases. The frequency equation of love waves is obtained from these solutions and the boundary conditions (denoting $\tau = \tau_{yz} = \mu\,(dv/dz)$, where we omit the factor $\exp\{ik(x-ct)\}$ are

$$\left.\begin{aligned}\frac{\dot v_{m-1}}{ic} &= \frac{\dot v_m}{ic}\\ \tau_{m-1} &= \tau_m\end{aligned}\right\}\text{on the } (m-1)\text{th boundray} \tag{37}$$

as well as for the existence of free surface waves,

$$\tau_1 = 0,$$
$$v_n \to 0 \text{ as } z \to \infty \tag{38}$$

Let $V_{m,1}$ and $V_{m,2}$ be the two linearly independent solutions as obtained in cases I to IV. Let the general solution of the mth layer can be written as :

$$V_m = C_{m,1} V_{m,1} + C_{m,2} V_{m,2} \tag{39}$$

where $C_{m,1}$ and $C_{m,2}$ are constants. Omitting the factors of x and t we get :

$$\dot v_m/ic = k\,(C_{m,1} V_{m,1} + C_{m,2} V_{m,2}) \quad \text{and}$$

$$\tau_m = C_{m,1}\,\mu_m(dV_{m,1}/dz) + C_{m,2}\,\mu_m(dV_{m,2}/dz) \tag{40}$$

the above equations can be expressed in matrix form as

$$P_m = K_m(z)\,A_m \tag{41}$$

where,

$$P_m = \begin{bmatrix}\dfrac{v_m}{ic}\\ \tau_m\end{bmatrix},\quad K_m = \begin{bmatrix} k\,V_{m,1} & k\,V_{m,2}\\ \mu_m \dfrac{dV_{m,1}}{dz} & \mu \dfrac{dV_{m,2}}{dz}\end{bmatrix}$$

$$\text{and}\quad A_m = \begin{bmatrix}C_{m,1}\\ C_{m,2}\end{bmatrix} \tag{42}$$

The boundary conditions, (37) can be written as

$$P_{m-1}(z) = P_m(z) \tag{43}$$

On the mth boundary ;

$$P_1(0) = \begin{bmatrix} -\dfrac{v_1(0)}{ic} \\ 0 \end{bmatrix}$$

and

$$A_n = \begin{bmatrix} C_{n-1} \\ 0 \end{bmatrix} \qquad (44)$$

assuming that

$$V_{n-1} \to 0,\ V_{n-2} \to \infty \text{ as } z \to$$

Placing the origin on the (m-1)th boundary we have from the equations

$$P_m(0) = K_m(0)\, A_m \qquad (45)$$

Setting $z = d_m$ (where d_m is the thickness of the mth layer) in the equation (5) we have on the mth interface

$$P_m(d_m) = K_m(d_m)\, A_m \qquad (46)$$

Eliminating A_m between equations (45) and (46) we get

$$P_m(d_m) = b_m\, P_m(0) \qquad (47)$$

where,

$$b_m = K_m(d_m)\, K_m^{-1}(0) \qquad (48)$$

Equation (47) gives a relation between the stress and displacement at the top and bottom of the mth layer. By repeated application of (47) and using the boundary condition (43) the relation between $P_n(0)$ and $P_1(0)$ is found to be

$$P_n(0) = b_{n-1} \ldots\ldots\ldots\ldots b_1 P_1(0) \quad (49)$$

Again since $P_n(0) = K_n(0)\, A_n$ (50)

We have, $K_n(0) = A_n = NP_1(0)$ (51)

where,

$$N = b_{u-1} \ldots\ldots\ldots\ldots b_1.$$

The equation can be written in component form using relation (44) as

$$k\, C_{n-1} V_{n,1}(0) = N_{11} \frac{\dot{v}_1(0)}{ic} \qquad (52)$$

$$C_{n,1}\, \mu_n(0) \frac{d V_{n,1}(0)}{dz} = N_{21} \frac{\dot{v}_1(0)}{ic} \qquad (53)$$

Dividing both sides of the equations (52) and (53) and thus eliminating $C_{n,1}$ and $\dot{v}_1(0)$ we get the love wave dispersion equation as

$$N_{21} = \frac{\mu_n(0)}{kV_{n,1}(0)} \cdot \frac{d_{n,1}(0)}{dz} N_{11} \qquad (54)$$

In particular, if the lowest semi-infinite medium (Fig. 1) is homogeneous, the dispersion equation is

$$N_{21} + \mu_n \sqrt{1 - c^2/\beta_n^2}\, N_{11} = 0 \qquad (55)$$

3, Discussion

The problem of computations of dispersion curves for different types of structures is under progress. Attempts are being made to chalk-out a computer programme using double precision to derive different phase and group velocity curves. The aim is being to fit dispersion data with a plausible structure. The results will be published separately.

The dispersion equation (55) derived in the foregoing analysis immediately brings out the results of Love (1911) and Stonely and Tillotson (1928) for a two and three layered structure of the earth respectively. Since the aim of the present paper is to present a mathematical analysis of the problem, no further conclusion is drawn at present until physical phase and group velocity curves are drawn and matched with a structure.

REFERENCES

Alramowitz, M. and Stegun, T. A. 1965 *Handbook of mathematical functions*, Dover publ.

Anderson, D. L. 1964 *Bull. seismol. Soc. Am.*, **54**, pp. 687-726.

Avtar, P. 1967 *Pur. appl. Geophys.*, **66**, pp. 48-68.

Bateman, H. 1928 *Bull. Am. math. Soc.*, **34**, pp. 343-348.

Biswas, N. N. and Knopoff, L. 1970 *Bull. seismol. Soc. Am.*, **60**, pp. 1123-1137.

REFERENCES (contd)

Bhattacharya, S. N. 1970 *Geophys. J.R. astro. Soc.*, **19**, pp. 335-349.

1972 Ph. D. Thesis, Delhi Univ., Delhi.

Chatterjee, S. N. 1969 *Indian J. Met. Geophys.*, **20**, pp. 349-356.

1971 *Geophys. J. astro. Soc.*, **23**, pp. 129-138.

Dorman, J., Ewing, M. and Oliver, J. 1960 *Bull. seismol. Soc. Am.*, **50**, pp. 87-175.

Ewing, W. M., Jordetzky, W. S. and Press, F. 1957 *Elastic waves in layered media*, Mc.Graw Hill Book Company, Inc., New York.

Haskell, N. 1953 *Bull. seismol. Soc. Am.*, **43**, pp. 17-34.

Jeffreys, H. 1928 *Mon. Not. R, astro. Soc*, Geophys. Suppl., **2**, pp. 101-111.

Kailis-Borok, V. I., Neigaus, M. G. and Shkadinskaya, G. V. 1965 *Rev. Geophys.*, **9**, pp. 105-109.

Mal, A. K. 1962 *Gerl. Brit. Zur. Geophys.*, **72**, pp. 82-88.

Maulick, T. N. 1965 *Bull. seismol. Soc. Am.*, **55**, pp. 121-133.

Matuzawa, T. 1929 *Bull. Earthq. Res. Inst.* (Tokyo), **30**, pp. 225-228.

Meissner, E. 1921 Vieteljahresschr. Naturfarseh, Ges. Zurich, **66**, pp, 181-195.

Press, F., Harkrider, D. and Seafeldt, C. A. 1961 *Bull. seismol. Soc. Am.*, **51**, pp. 495-502.

Randall, M. J. 1967 *Ibid.*, **57**, pp. 1299-1316.

Saastamoninen, P. 1969 *Geophyica*, **10**, pp. 35-43.

Satô, Y. 1952 *Bull. Earthq. Res. Inst. Tokyo*, **30**, pp. 1-12.

Sinha, N. K. 1969 *Pur. appl. Geophys.*, **73**, pp. 47-59.

Stoneby, R. and Tillotson, E. 1928 *Mon. Not. R. astro. Soc.*, Geophys. Suppl., **1**, p. 521.

Thomson, W. 1950 *J. appl. Phys.*, **21**, pp. 89-93.

Thrower, E. N. 1965 *J. Sound Vibrations*, **2**, pp. 210-216.

Vlaar, N. J. 1966 *Bull, seismol. Soc. Am.*, **56**, pp. 1305-1315.

Watson, T. H. 1970 *Ibid.*, **60**, pp. 161-166.