Indian J. Met. Hydrol. Geophys. (1977), **28**, **4**, 465-466

551·511

On the theory of the boundary layer

LEON COVEZ

15 Av. de la Porte d' Asnieres 75017, Paris, France

(Received 21 March 1977)

ABSTRACT. The problem of the diurnal wind variation inside the boundary layer is treated using the diffusion equation with a tensorial diffusion coefficient which is more adequate than the ordinary way of dealing with this question in terms of a 'fictitious viscosity' coefficient. The correlation tensor is expressed as a random function and using the equivalence between the diffusion and the wave equation some characteristics of the solution are given. In particular with the hypothesis of a fundamental periodicity of 24 hr. for the correlation function, the classical elliptical variation of the wind hodograph is obtained. An analogy is established between the general theory of relativity with its use of Riemann spaces and the non-euclidean frame necessary for the study of the movements in the lower layer due to the functional relation among space and time variations of diffused quantities. The study is restricted to the case of homogeneous turbulence and this excludes the application to the surface layer. The mathematical basis are the papers of Varadhan and Zauderer concerning the behaviour of the asymptotic solutions of the heat and wave equations.

1. Introduction

One of the most fundamental properties of the boundary layer, are the changes in magnitude and direction of the wind with the height. Those changes begin about the level of 10-20 m up to 1 km in the middle latitudes of the earth. Below 10-20 m the direction of the wind is constant and the speed changes according to a logarithmic law.

This problem has been studied in particular by the Soviet school of meteorology starting with the equations of motion for the atmosphere and introducing a turbulent exchange coefficient that is supposed to be variable in time and space. This coefficient can be looked upon as a kinematic coefficient of viscosity which is a rather gross approximation since in fact it is a correlation tensor.

In what follows we shall use a diffusion equation for the momentum and study the asymptotic behaviour for small intervals of time.

2. Mathematical Model

If $\mathbf{q} = \rho\mathbf{v}$ is the momentum of a fluid element, the mean concentration of $\mathbf{q}$, $\mathbf{P}(\mathbf{r}, t)$ satisfies the following differential equation (Batchelor 1949):

$$\frac{\partial \mathrm{P}(\mathbf{r}, t)}{\partial t} = K_{ij} \frac{\partial^2 \mathrm{P}(\mathbf{r}, t)}{\partial x_i \, \partial x_j} \tag{1}$$

where K_{ij} is the diffusion coefficient.

We must add the boundary conditions

$$\lim_{t \to 0} \mathrm{P}(\mathbf{r}, t) = j + \text{boundary condtitions}$$

Eq. (1) is the equivalent of the Helmholtz Eq. (Frisch 1969) that is,

$$\nabla^2 \Psi(k) + k^2 n^2(\mathbf{r}, t) \Psi(k) = j \tag{2}$$

where

k = wave number

Ψ = wave function

n = refraction index

We are interested in studying the behaviour of Ψ when $k \to \infty$ that is for movements at the small scales. The asymptotic behaviour of (1) and (2) for $t \to 0$ and $k \to \infty$ have been studied by Varadhan (1967) and Zauderer (1970). The coefficients k_{ij} in (1) determine a Reimaniann metric with a length invariant $d^2(\mathbf{r}, t)$. It is possible to show (Varadhan 1967) that

$$\lim_{t \to 0} [-2t \log \mathrm{P}(\mathbf{r}, t)] = d^2(\mathbf{r}, t) \tag{3}$$

3. Analysis of the asymptotic approximation

We suppose that k_{ij} is a random variable analytic and stationary. After Wehrle (1944) the most general correlation function in this case is

$$R(\tau, r) = \overline{\cos(\Omega\tau - Mr)} \qquad \tau = t_2 - t_1 \qquad r = |\mathbf{r}_2 - \mathbf{r}_1| \tag{4}$$

where,

the bar stands for the 'mean of' and Ω M and are two random numbers defined by a probability law

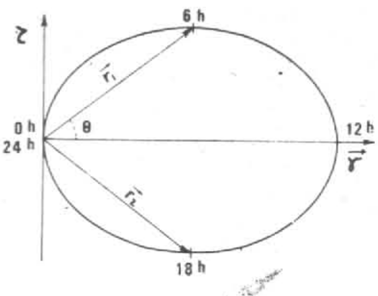

Fig. 1 The horizontal axis is parallel to the isobar at the ground, r1 *and* r2 are related, with the maximum heights where the wind is geostrophic,.

$f\ (m,\ m')$

We take Ω and M without too much dispersion, which is equivalent to linearize the problem and we introduce the following parameters :

$$m_\circ = \overline{\Omega} \qquad m_\circ' = \overline{M}$$
$$\sigma = \overline{\Omega' M'} \qquad \sigma' = \overline{M^2}$$
$$h = \overline{\Omega'^2}$$

The quadratic invariant form that corresponds to Eq. (4) is (Wehrle 1944) :

$$d^2(\mathbf{r}, t) = \sigma^2 r^2 - 2m_\circ m_\circ' r \tau + \sigma'^2 \tau^2 \qquad (5)$$

the characteristic Eq. $d^2(\mathbf{r}, \mathbf{t}) = 0$ represents a conic which can be :

$$(m_\circ m_\circ')^2 - 4\sigma^2 \sigma'^2 \geqslant 0 \quad \begin{matrix} \text{hyperbola} \\ \text{parabola} \\ \leqslant \ \ \text{ellipse} \end{matrix}$$

In our case $m_\circ \sim \sigma$ and $m_\circ' \sim \sigma'$
then $(m_\circ m_\circ')^2 - 4\sigma^2 \sigma'^2 < 0$

This means a transfer of momentum or energy where the propagation surface has the properties of an ellipsoid of revolution.

Taking for the main period of (4) $24h\mathbf{r}$, we have for $d^2(\mathbf{r}, \mathbf{t})$ the geometric Fig. 1.

The fundamental solution of Eq. (1) gives the probabilities of transition of the diffusion process $\mathbf{X}(\tau)$ associated with the equation and the asymptotic approximation (3) gives the behaviour of $X(\tau)$ when the time intervals are small.

Let us see Fig. 1, the vector $\mathbf{r}$ gives the magnitude of $\log P(\mathbf{r}, t)$ and it is easy to obtain the qualitative behaviour of the changes of $\mathbf{v}$ in magnitude and direction.

We see that the changes of the wind in time and space are not independent. Concerning the magnitude of $\mathbf{v}$ there is a minimum towards $24h$, $0h$ and a maximum towards $12h$ (it seems infinite due to the linearized hypothesis).

If we take the horizontal axis parallel to the isobars we get for $6h$—$18h$ a wind parallel to the isobars, that is geostrophic. Afternoon there is a reversal of direction.

4. Conclusions

We think we have shown that the main characteristics of the diurnal wind variation can be obtained from a diffusion equation. We have taken into account turbulence,introducing as diffusion coefficient a random function which defines the metric of a Reimaniann space. This fact, limit us to the case of homogenuous turbulence and the validity of Eq.(1) is subject to caution for the layers too near to the ground. Any other differential equation has the same limitations since the mathematical techniques are not yet developed to deal with the general problem.

Any how we can conclude that the movement in the lower layers should not be studied in the frame of Euclidean space where the variables of space and time are independents problems of boundary layers, vortex motion and in general turbulence are closely related to the mathematical framework built by the physical theory of relativity.

REFERENCES

Batchelor, G.K.	1949	Diffusion in a field of homogeneous turbulence, *Aust. J. Sci. Res.* **2** 437-450.
Belinskii, V. A.	1961	*Dynamic Meteorology*, Israel Program for scientific translations, Jerusalem, p. 416.
Frisch, U. and Filippi, P.	1969	*C. R. Acad. Sci. Paris*, **268**, 804-807.
Varadhan, S.	1967	*Comm. Pure Appl. Math.*, **20**, 431-455.
Wehrle, P.H, and Dedebant G.	1944	Le calcul aleatoire, 107-116., Portugaliae Physica Lisbonne.
Zauderer, E.	1970	*J. Math. Anal. Appl.* **30**, 157-171.