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A finite element model for the study of heated island effects

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सार — भूमंडलीय परिसीमा स्तर प्रवाहशासी अरैखिक समीकरणों की स्थिर दशाहल को सामान्य परिमित-अन्तर तकनीक के अतिरिक्त परिमित तत्व विधि से इंगत किया गया है। भौतिक परिघटना एक ही दशा में सतह के खुरदरेपन और तापमान पर से प्रवाह से संबंधित है। परिणामों की तुलना अपनाई गई दोनों तकनिकों में काफी सामंजस्य प्रदर्शित करती है। दोनों दिशाओं में विचरणों और अस्थिर मामलों को सम्मिलित करने के लिए सामान्यीकरण काम चल रहा है।

ABSTRACT. Steady state solutions for the nonlinear equations which govern planetary boundary layer flow is obtained through finite element method as well as by the usual finite difference technique. The physical phenomenon corresponds to that of a flowover a surface varying in surface roughness and temperature along one direction. Comparison of the results show a good agreement between the two techniques employed.

1. Introduction

With the rapid development of digital computers the partial differential equations occurring in various fields of science and technology, hitherto unamenable to get a closed analytical solutions, are solved using various numerical techniques especially using the finite differences technique. However, other engineering fields, especially aircraft technology, in which complex partial differential equation arises has used the finite element method to obtain solutions. The purpose of this paper is to develop a finite element model for the study of heated island effects using the relevant planetary boundary layer (PBL) equations. For comparison purposes a finite difference model is also developed. Due to differential heating of land and surrounding water mass or cooler areas, islands generate local circulations of mesoscale proportions. The interchange of heat energy, momentum and moisture between air and sea combine with the variational properties such as terrain roughness, temperature, size and contour features to produce a variety of atmospheric circulations. The nature of the induced perturbations depend upon the ambient wind or the large scale flow and the degree of stratification of the atmosphere in addition to the above mentioned factors. Under suitable conditions, the cloud formation and precipitation may occur.

Theoretical studies of flow over heated surfaces have been made by Malkus and Stern (1953), Smith (1957), Taylor (1969) to mention a few. An extensive study incorporating non-linear effect has been done by Estoque and Bhumralkar (1970). However, in all the above studies either a semi-analytical approach and/or finite difference technique has been employed to obtain the desired solutions. In a general case, including nonlinear effects, an analytical study encompassing the desired parameters is not feasible. Hence a numerical technique, in this paper finite element method, is employed to obtain the desired solutions.

This kind of study is of interest to India as it is conjectured that many off-shore islands may have a considerable effect in the enhancement of the rainfall along coastal regions. The final objective of this study is to investigate the quantitive changes in rainfall amount due to presence of off-shore islands. For this, detailed PBL observations are necessary. Hence, in the absence of observational data, we present in this preliminary study, the solutions for a homogeneous terrain, *i.e.*, homogeneous with respect to temperature, roughness etc using this as our first approximation, solutions for flow over a one dimensionally varying terrain under steady state conditions are obtained. The procedure is to get a first approximation in simple cases and this solution is improved by introducing successively the variations of parameters such as temperatures or terrain roughness to study their effects. The equations are solved using Galerkin residual procedure, an extension of Rayleigh-Ritz principle, instead of usual finite difference technique.

2. Finite element procedure vis-a-vis **finite difference for the present problem**

Better or finer resolution for the dependent variables at any particular region is obtained by varying the size/shape of the finite element conveniently. In the finite difference technique finer resolution in the region where the variables have a large gradient is obtained by varying grid size by logarithmic transformation of a coordinate and/or similar other transformations. This is usually a tedious process. Moreover, the irregular geometry of island can be better approximated with the finite element technique rather than in the finite difference grid system. However, it has been found in practice that the finite element technique require more computational time than that of finite difference technique. In this study, both finite difference and finite element techniques were employed to check and verify the correctness of calculations.

3. Flow over one dimensionally varying terrain

The appropriate governing equations, neglecting the variations along y-direction are the following:

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(K\frac{\partial u}{z}\right) \quad (1)$$

$$v\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) \quad (2)$$

$$u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial z}=\frac{\partial}{\partial z}\left(K\frac{\partial \theta}{\partial z}\right) \quad (3)$$

$$u\frac{\partial Q}{\partial x}+w\frac{\partial Q}{\partial z}=\frac{\partial}{\partial z}\left(K\frac{\partial Q}{\partial z}\right) \quad (4)$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad (5)$$

$$\frac{\partial}{\partial z}\left(\frac{p}{p_0}\right)^k=-\frac{kg}{R\theta}, \quad \left(k=\frac{R}{c_p}\right) \quad (6)$$

The last two equations are the equation of continuity and the hydrostatic relation respectively. In these equations, R is gas constant for air, c_p is specific heat at constant pressure, g is gravity and p_0 is 1000 mb. The horizontal pressure gradient $\frac{1}{\rho}\frac{\partial p}{\partial y}$ is to be prescribed by the large scale geostrophic flow. The x component $\frac{1}{\rho}\frac{\partial p}{\partial x}$ except at the top of boundary layer is to be calculated using the hydrostatic relation at each iteration step.

The vertical distribution of wind, temperature and moisture over a homogeneous terrain can be obained from the above differential equations by neglecting non-linear advective terms.

By specifying appropriately the eddy coefficient K (assumed to be the same for heat, momentum and moisture transfer) we can obtain analytical solutions. With appropriate boundary conditions and for constant K the above equations yield the well known Ekman relations. However, in the real atmosphere, the eddy coefficient K is a function of altitude, stability parameter, roughness length and the shear of basic wind. The specification of K including the above mentioned parameters is not unique. For our calculation, we have used Blackadar (1962) formulation as above

$$K=\begin{cases} l^2\frac{\partial V}{\partial z}\,(1+\alpha R_i) & R_i>0 \\ l^2\frac{\partial V}{\partial z}\,(1-\alpha R_i)^{-1} & R_i\leqslant 0 \end{cases} \quad (7)$$

where $l=k_0\,(z+z_0)\Big/\left(1+\frac{k_0\,(z+z_0)}{\lambda}\right)$,

$\lambda=2.7\times10^{-4}\,U_g/f$,

$U_g=(u_g{}^2+v_g{}^2)^{1/2}$, $k_0=0.4$,

$\alpha=-3.0$ and

$z_0=0.1$ (Roughness length) is varied.

Since the eddy coefficient is assumed to be positive, the absolute value of $\partial V/\partial z$ is used. However, we have found in our experience, in order to have a smooth variation of K over the entire domain, $\partial V/\partial z$ should be replaced by the expression :

$$\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{1}{2}}$$

The scientific reason for using the above expression is also given by Torrance *et al.* (1973).

4. Method of solution

For the simple case of homogeneous terrain with K constant, Galerkin's technique for weighted residuals yield (for the first two equations) the following relations :

$$\int N_i\left(K\,\frac{\partial^2 u}{\partial z^2}+f\,v\right)dz=0 \quad (8)$$

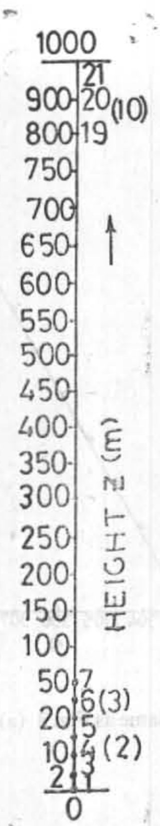

Fig. 1. Structure of the finite element assembly with nodes used for homogeneous terrain calculations

$$\int N_i \left(K \frac{\partial^2 v}{\partial z^2} - f u + f u_g\right) dz = 0 \quad (9)$$

where N_i is the test function. The dependent variables, u, v are now represented over a particular finite element by the following expressions :

$$u = \sum_{j=1}^{3} N_j u_j, \quad v = \sum_{j=1}^{3} N_j v_j \quad (10)$$

where N_j represents a piecewise continuous function which in this case is also the test function. Inserting the above expressions into the integrals, integrating by parts a system of N equations with N unknowns are generated at the nodal points. Thus, for the entire domain under consideration we can write the basic equations in a compact form :

$$[F]\,[V] = [D] \quad (11)$$

where $[F]$ is a matrix consisting of the basic functions and their derivatives over z, $[D]$ and $[V]$ are column vectors consisting of constants and unknown variables u and v. The imposed boundary conditions are included in the column vector D. We have imposed the following boundary conditions for the problems :

At $z = 0 : u = 0,\ v = 0,\ \theta = \theta_0,\ Q = Q_0$

At $z = H : u = u_g,\ v = v_g = 0,\ \theta = \theta_H,\ Q = Q_H$

where H is the height of planetary boundary layer,

$fu_g = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial v}$ and f is coriolis parameter.

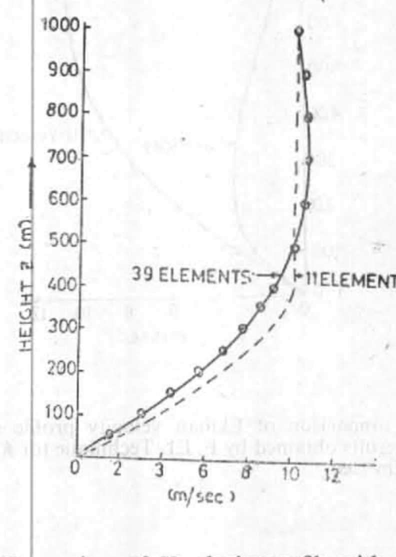

Fig. 2. Comparison of U velocity profile with the change in the number of elements, for $K = 5 \times 10^4$ cm²/sec

In a similar manner, now specifying the eddy coefficient K using the expression of Blackadar (1962) we can reduce the basic equations to a matrix form. By solving this matrix equation, we obtain the distribution of variables in the entire domain. Since K is a variable and function of z the matrix coefficients containing K should be evaluated at all nodal points in order to include shear of basic wind and stability parameter. The same test function (N_i) was used for K in this analysis.

An identical approach is made use of to evaluate temperature and humidity profile in the entire domain. With K as constant, the temperature and humidity equations can be solved separately. However, with K as a function of stability parameter and shear of basic wind for each iteration u, v components are evaluated first and substituting these in expression for K temperature and humidity equations are solved. The Richardson number is assumed to be constant with height and equal to the average value between surface and 100 metres.

For computational purposes, it is necessary to specify the values of the variables at the boundaries, *i.e.* at

$z = 0$ and $z = H$ (Top of PBL)

The assumed typical values are :

At $z=0,\ u=0,\ v=0,\ \theta=303°$A, $Q=14$g/kg

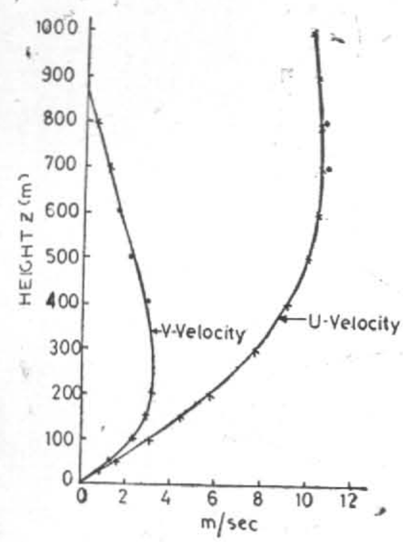

Fig. 3 (a). Comparison of Ekman velocity profile swith the results obtained by F. El. Technique for $K=5\times10^4$ cm²/sec

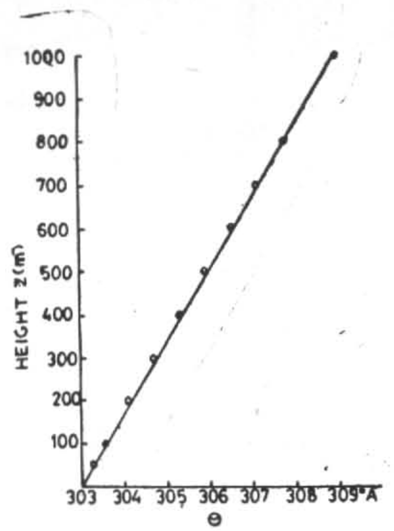

Fig. 3 (b). Same as Fig. 3 (a), Temp. Profile

At $z=H$, $u=1000$ cm sec^{-1}, $v=0$, $\theta=309$°A $Q=12$g/kg.

At $z=H$, u and v specify simply the values of geostrophic wind. For simplicity, it is assumed $v=0$ at $z=H$.

Two typical values for z_0 (roughness length) are chosen ;

$z_0=0.1$ cm (smooth and flat)

$z_0=100$ cm (tall grass)

Similarly the temperature at the bottom surface ($z=0$) is specified either as $\theta=303$°A or $\theta=313$° A, the later corresponding to a heated island surface. To test the validity of finite element technique few runs were made with different values of K which can vary over large range. By varying the size of finite element the accuracy of adopted technique is tested.

Typical results are given in accompanying figures. Fig. 1 shows the structure of line elements in the domain considered. Fig. 2 shows the increase in accuracy of calculations with increase in mesh size for $K=5\times10^4$ cm²/sec. For comparison purposes results obtained using analytical formulae are also shown.

The temperature and humidity profile obtained with a typical value of K are shown in Fig. 3. The results obtained using finite element technique and finite difference method is compared in Fig. 4 for variable K.

To obtain solutions for flows with variation in surface roughness and/or temperature the region is divided into rectanguar elements.

The structure of the integrating domain, discretization into rectangular elements with nodes at the corners, is shown in Fig. 5. The grid system for integrations has the coordinates along z axis shown in Table 1. Also shown the grid distances along x-axis. The integrating (Gaussian) point for each element is taken at the centre for each element. Our efforts with two point Gaussian integration which may be more accurate have not been quite satisfactory. Thus, the results are for with one point Gaussian integration only with 19 elements in each column.

The following boundary conditions, similar to that of Estoque and Bhumralkar (1970) prescribed: at the upwind lateral boundary :

$$x=0: \ u=u(0,\ z), \quad v=v(0,\ z), \quad \theta=\theta(0,\ z),$$

$$Q=Q(0,\ z)$$

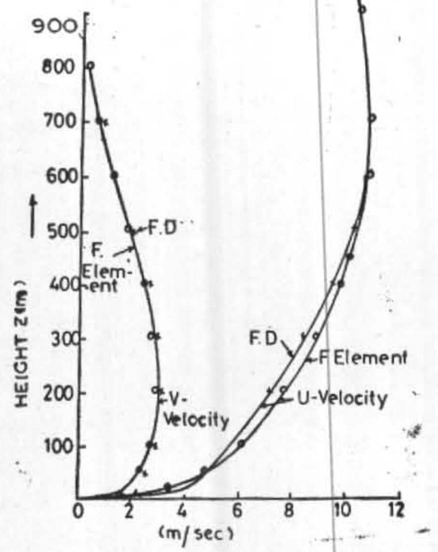

Fig. 4. F. Element and F. Diff. results for homogeneous case, $z_0 = 0.1$ cm

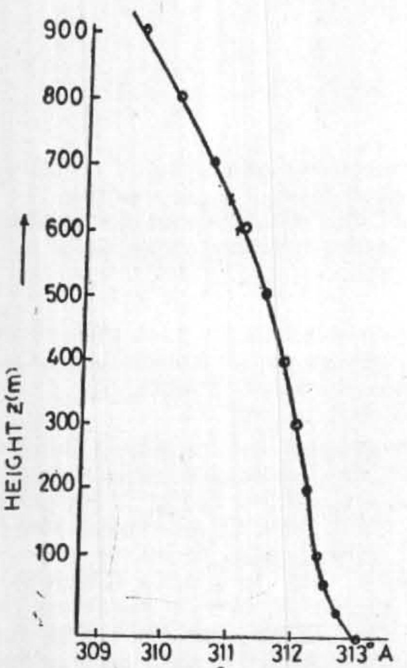

Fig. 4 (a). Same as Fig. 4, Temp. profile

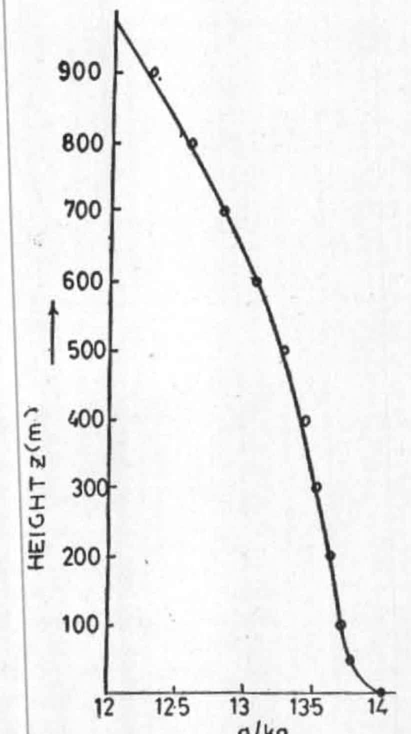

Fig. 4 (b). Same as Fig. 4, Humidity profile

at the lower boundary :

$z=0$; $u=0$, $v=0$, $\theta=\theta(x, 0)$, $Q=Q(x, 0)$,

at the upper boundary :

$z=H$; $u=u_g(H)$, $v=v_g(H)$, $\theta=\theta(H)$, $Q=Q(H)$

The values of variables at the top boundary, $z=H$, are taken indentical as given previously. For simplicity, constant upper boundary pressure $p(x, H)=900$ mb is prescribed. The vertical distributions of the variables at the upwind lateral boundary $x=0$ are taken to be that of homogeneous terrain values, obtained as described in previous section.

As before, the variables are presented over a particular finite element, in this case over a rectangle, by a linear combination of basic functions such as :

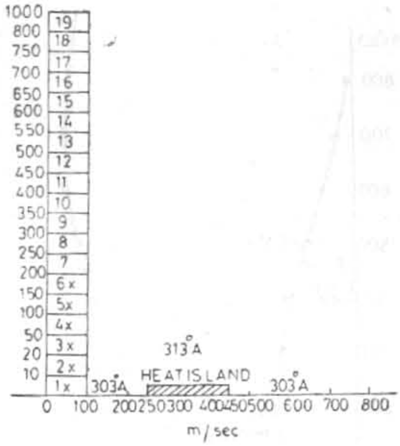

Fig. 5. Finite element assembly used for one dimensional variation calculations

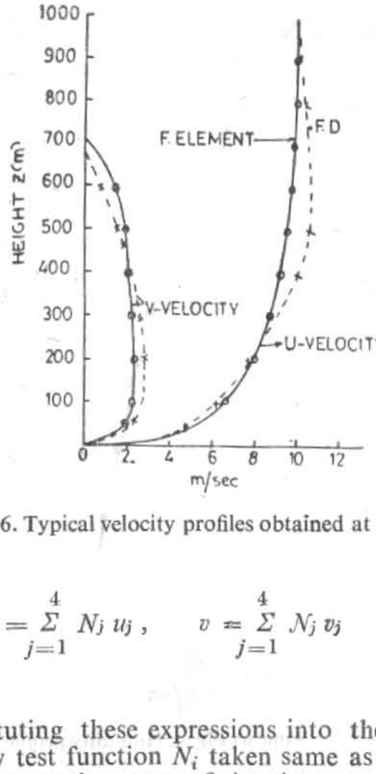

Fig. 6. Typical velocity profiles obtained at $x=100$ m

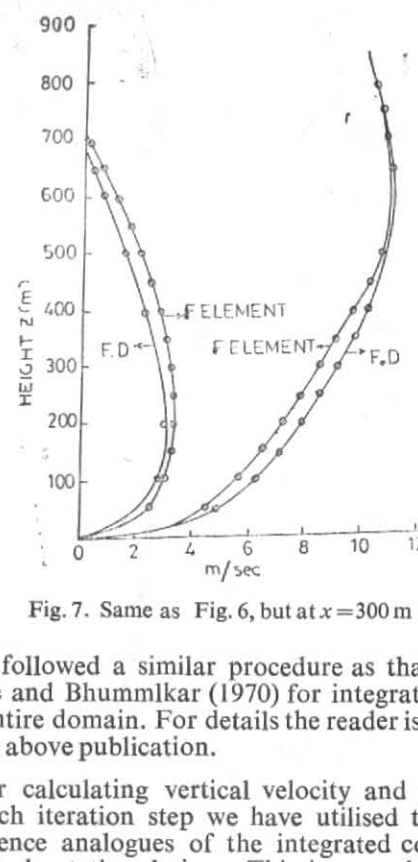

Fig. 7. Same as Fig. 6, but at $x=300$ m

$$u = \sum_{j=1}^{4} N_j u_j , \qquad v = \sum_{j=1}^{4} N_j v_j \qquad \text{etc.}$$

substituting these expressions into the equations and by test function N_i taken same as basis function, integrating over a finite element and assembling over the domain we obtain a set of N equations with N unknowns, N being the number of nodal points. Recasting these equation assembly into a matrix form and solving, we obtain solutions of these variables at each node, subjected to the imposed boundary conditions. As before, few runs were made to determine the optimum number elements necessary for the desired accuracy. We have followed a similar procedure as that of Estoque and Bhummlkar (1970) for integrating over the entire domain. For details the reader is referred to the above publication.

For calculating vertical velocity and pressure at each iteration step we have utilised the finite difference analogues of the integrated continuity and hydrostatic relations. This is essentially done to save the computational time. The results obtained by both the techniques are compared in Figs. 6-8. Problems of computational instability which has necessiated the use of upstream space differencing with finite differencing scheme has not risen with finite element technique. Further refinements in the nodal may necessiate this. The results obtained show finite element solutions has a smoother

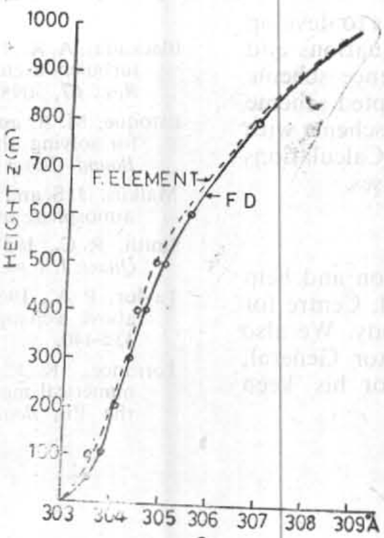

Fig. 8. Temperature profiles at $x = 100$ metres

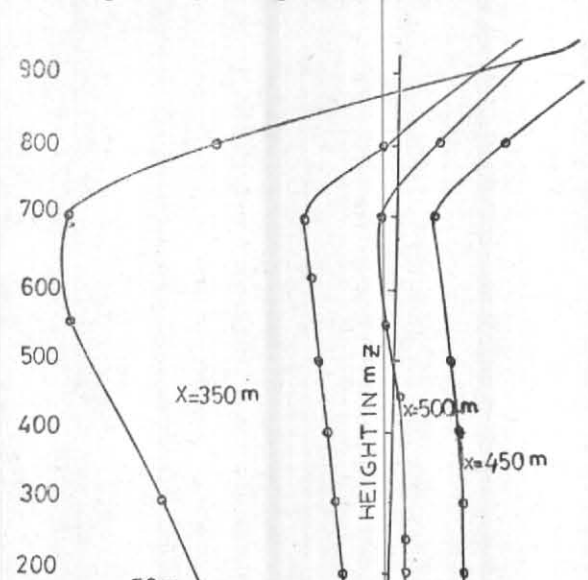

Fig. 9. Vertical velocity profiles at various distances downwind along x—axis

variation and has a better accuracy than those of finite difference technique.

The vertical velocity profiles obtained at various distances downwind along the x-axis are shown in Fig. 9. It is clear from this study that a convective cell develops at the edge of heated island in the down wind direction rather than over the island itself. Hence, precipitation can occur only at the down wind edge of the island. This is in agreement with the observations as reported by Malkus and Stern (1953).

Thus the results show the feasibility of adopting the finite element scheme to a study of PBL equations and in some cases better accuracy in results can be obtained than finite difference method.

5. Conclusion

The primary objective of this paper is to develop finite element scheme to solve PBL equations and compare the results with finite difference scheme results. A brief description of the adopted scheme is given above. The application of the scheme with real data used as input is to be awaited. Calculations are in progress for studying unsteady cases.

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