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Basic approximations for large scale motion in equatorial regions

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ABSTRACT. An exploratory study of approximations for the equatorial region is given to stress the rapid variation of coriolis parameter with latitude, and the consequential differences in the length scales in meridional and zonal directions. The resulting equations retain coriolis as well as non-linear inertia terms in zonal momentum balance. This approximation for dominantly zonal flow is supplemented with another for a sublayer near the equator where the meridional flow might be dominant in case of large interhemispheric transport, as in monsoon over the Indian Ocean.

1. Introduction

Approximations built around geostrophic motion, such as the well known quasi-geostrophic approximation (Charney 1955), have proved to be of considerable utility in providing an analytical framework for numerical and theoretical studies. While detailed scale analysis (Phillips 1963) is available to provide a theoretical basis for these approximations for middle latitudes and polar regions, the situation is quite different for equatorial and tropical regions. Here, there are hardly any scale analysis arguments available and doubts have been expressed about the validity of these approximations in low latitudes (Phillips 1963).

Simple estimates throw light on the possible applicability of well known approximations to low latitudes. Rossby number (C/fL) based on a velocity scale C of, say, 15 m sec^{-1}, a length scale L of 10^3 km, and coriolis parameter f changes from 0·15 at a latitude of 45° to 0·59 at a latitude of 10°, and of course, it becomes infinite at the equator. Thus we have the possibility that, in the low latitudes, the non-linear inertia terms may not be as small as required by the geostrophic or related approximations. Such a behaviour is expected essentially in a rather limited range of latitudes around the equator. There are many instances in fluid mechanics (*e.g.*, van Dyke 1975) where an approximation applicable over a wide domain breaks down in a small region. Boundary layers and shocks are classic examples of such regions. These regions are often thin and the length scale in one direction is of an order of magnitude smaller than the scale in another. Very rapid variation in coriolis parameter with latitude near the equator permits the length scale for the meridional direction of being much smaller than that in the zonal direction. This type of very rapid variation of physical quantities in one direction compared to that in another makes certain terms in the basic equations, and of course corresponding forces, important in the thin regions, although they may be quite unimportant outside the thin region. We examine in the present paper whether the region around equator, having a rather limited range of latitude, can be analysed by techniques which have worked satisfactorily elsewhere (van Dyke 1975).

The role and potential utility of approximations for low latitudes can probably be better judged subsequently, but general experience with a approximations in fluid mechanics provides a few pointers. The approximations provide a measure of understanding of these special regions by stressing the role of certain force terms which are critically important in the regions. The resulting simplifications of equations are often so great that they open up possibilities of analytical and numerical studies of a host of phenomena. In addition, such understanding could also be utilised in devising numerical schemes for larger domains, in case the requirements of computer memory and time for straightforward numerical integration of complete equations are beyond the available computer capacity. A more specific and direct indicator is that low-latitude inviscid approximation would provide an appropriate starting point for a study of equatorial planetary boundary layer.

This paper explores approximations designed for low latitudes. The emphasis here is on the examination of the main issues with relatively simple mathematics. It is felt that if a detailed scale analysis is attempted in the same style and spirit as for the well-developed case of middle latitudes (Phillips 1963), the algebra would unduly complicate matters. Hence such a more comprehensive examination involving the effects of spherical geometry, orography, unsteadiness, frictional forces, and even the horizontal divergence associated with heating is postponed to a subsequent stage. Also, the analysis relies implicitly on the ideas of scale analysis as applied to meteorology, those of matched asymptotic expansions (van Dyke 1975), and of boundary-layer theory (Schlichting 1968), although the treatment is kept as informal and simple as possible.

2. Basic scale analysis

There are two basic ways of developing approximations of the present type. One is to start with estimates of orders of magnitude of various quantities and then to draw conclusions about the consequences on the equations. Another is to introduce a limited number of non-dimensional quantities which govern the behaviour of orders of magnitude of various terms. Then we examine possible approximations depending on assumptions of non-dimensional quantities. While the former method has the advantage of directness, the latter exploits the nature of the equations and reduces the required assumptions. We follow here the latter method.

We start with the equations for steady inviscid motion free of horizontal divergence based on β-plane approximation.

$$uu_x + vu_y - fv = -\hat{\phi}_x$$

$$uv_x + vv_y + fu = -\hat{\phi}_y$$

$$u_x + v_y = 0 \qquad (1)$$

where u and v are velocity components in the zonal and meridional directions x and y, $\hat{\phi}$ is the geopotential and f is the coriolis factor. The suffixes x and y indicate partial differentiation keeping pressure p fixed.

Let L and l denote length scales in the zonal and meridional directions and U and V denote corresponding velocity scales. Fig. 1 shows the streamlines and isotach lines for mean monthly airflow at 1 km for July in the Indian Ocean. It is noticed that, barring certain locations, zonal velocity component by and large dominates the meridional velocity component. The streamlines in Fig. 1 however suggest that right at the equator the motion is largely meridional. We have two available options. One is to examine the case in which meridional velocity component is, by hypothesis, of an order smaller than the zonal component. Such a case would deal with low latitudes. The close vicinity of equator (a sublayer) has to be treated separately by a supplementary approximation in which the meridional motion dominates. The other option is to allow both components to be of the same order. We follow here the first option.

Let V/U be ε and l/L be δ. Further let $\hat{\phi}$ $(x,y,z,)$ be a z-dependent average $\bar{\phi}$ (z) *plus* a variable ϕ (x,y,z), which we take to be of order U^2. Orders of the terms of Eqn. (1) are then

$$U^2/L,\ (\varepsilon/\delta)\, U^2/L,\ (\varepsilon/R_0)\, U^2/L,\ U^2/L;$$

$$\varepsilon(U^2/L),\ (\varepsilon^2/\delta)\, U^2/L,\ (1/R_0)\, U^2/L,\ (l/\delta)\, U^2/L;$$

$$U/L,\ (\varepsilon/\delta)\, U/L,$$

where the Rossby number R_0 is U/fL. We take ε to be of the order δ so that the continuity equation retains its form. We set ε to be equal to δ, that

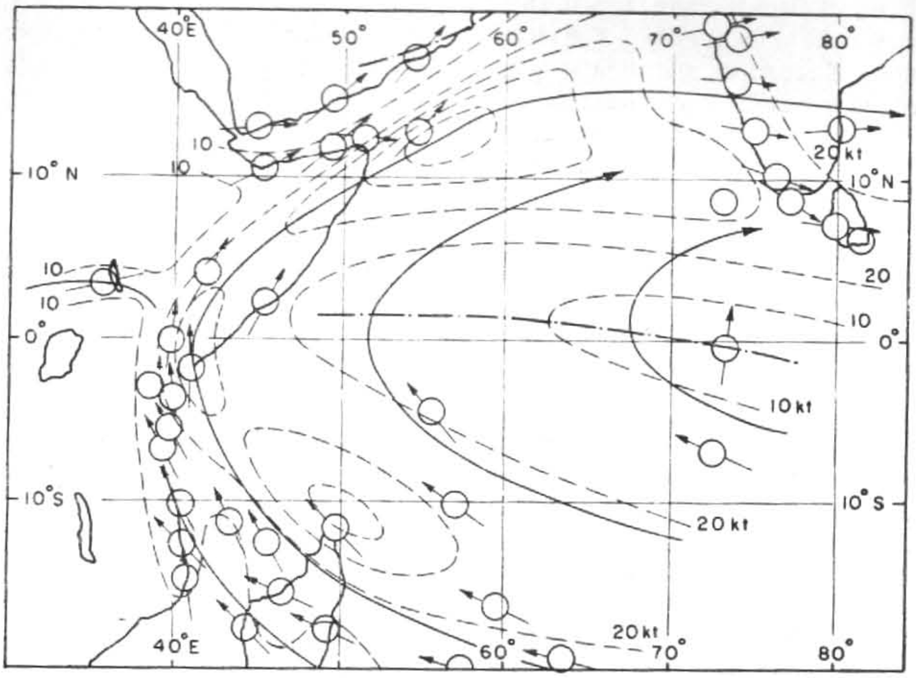

Fig. 1. Mean monthly air flow at 1 km — July (*From* Findlater 1971)
→ Streamlines, — — isotachs and —O→ observation stations

is, the ratio of velocity scales V/U is equal to the ratio of length scales l/L. Now we consider approximations in which coriolis and some non-linear inertia terms are retained. This requirement implies R_0 is of the order of ε. For low latitudes f is of the order of βl, so that R_0 is of order $(1/\delta)\ (U/\beta L^2)$. The condition is then $(U/\beta l^2)$ be of order unity. With U equal to 15 m sec^{-1}, β equal to $2.3\times10^{-11} m^{-1} sec^{-1}$ at the equator and l equal to say, 5×10^2 km, $(U/\beta l^2)$ becomes 2.6 which is certainly of order unity. If we allow ε to be, say, 0.15, which is taken as an acceptably small value of Rossby number taken for midlatitudes for purposes of scale analysis (Phillips 1963), the zonal length scale would be about 3.3×10^3 km, which is perhaps a bit large but not entirely unreasonable. Also, the meridional velocity scale V would then be 2.25 m sec^{-1}, which is also plausible.

When ε, δ and R_0 are of the same order, the equation reduces to

$$uu_x + vu_y - fv = -\phi_x,$$

$$fu = -\phi_y,$$

$$u_x + v_y = 0 \qquad (2)$$

Hence this approximation retains non-linear terms in the zonal balance but not in the meridional balance. So the meridional balance is still geostrophic. It is interesting to note these equations resemble boundary-layer equations despite the absence of diffusive terms.

The preceding equations can also be obtained by asymptotic expansions. There are a few approaches with small differences. One type is given below. It is applicable to a range of latitude where f can be approximated well be βy.

$$\psi(x,y,z;\varepsilon) = \varepsilon UL\ [\psi_0\ (X,\ Y) + \varepsilon\psi_1\ (X,Y) + \varepsilon^2\psi_2\,(Y,X)\ldots.]$$

$$\hat{\phi}\,(x,y,z;\varepsilon) = \overline{\phi}(Z) + U^2\,[\phi_0(X,Y) + \varepsilon\phi_1(X,Y) + \ldots]$$

$$f(y\,;\varepsilon) = \beta\varepsilon L\,[Y - \frac{\varepsilon^2}{6}\,(L/a)^2 Y^3 + \ldots] \qquad (3)$$

where a is the radius of the earth, and where we have not shown the dependence of terms in the square brackets on the vertical or pressure coordinate. Here X and Y are given by

$$X = x/L, \quad Y = y/\varepsilon L \tag{4a}$$

and the u and v components are given by

$$u = U\,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)$$
$$v = \varepsilon U\,(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots) \tag{4b}$$

where u_1 and v_1 are ψ_{iY} and $-\psi_{iX}$. The lowest order equations are then

$$u_0 u_{0X} + v_0 u_{0Y} - Y v_0 = -\phi_{0X}$$
$$Y u_0 = -\phi_{0Y}$$
$$u_{0X} + v_{0Y} = 0 \tag{5}$$

The above equations are essentially the same as (3) since the f term after expansion (4a) has the lowest order term proportional to Y, $(U/\beta L^2)$ being equal to ε^2.

The next order equations are

$$u_0 u_{1X} + u_1 u_{0X} + v_0 u_{1Y} + v_1 u_{0Y} - Y v_1 = -\phi_{1X}$$
$$Y u_1 = -\phi_{1Y}$$
$$u_{1X} + v_{1Y} = 0 \tag{6}$$

Thus we see that the non-linear inertial terms do not contribute to the meridional balance even in the first order.

3. Similarity solutions

We restrict our attention to the range of latitudes where f can be approximated by βy. The form of Eqn. (3) suggest similarity solutions of the following form

$$\psi = Ag^3\,(x)\,F(\eta), \quad \phi = Bg^4\,(x)\,G\,(\eta), f = \beta y,$$
$$\eta = y/g\,(x) \tag{7a}$$

where A and B are constants. The velocity components are then in the form

$$u = Ag^2\,(x)\,F'\,(\eta),\ v = Ag^2\,(x)\,g'\,(x)\,(\eta\,F' - 3F) \tag{7b}$$

The resulting ordinary differential equations for F and G are

$$(3\,FF'' - 2\,F'^2) + \frac{\beta}{A}\,\eta\,(\eta\,F' - 3\,F)$$
$$= -\frac{B}{A^2}\,(\eta\,G' - 4G)$$
$$\frac{\beta}{A}\,\eta F' = -\frac{B}{A^2}\,G' \tag{8}$$

On elimination of G, we get

$$3FF''' - F'F'' + \frac{\beta}{A}\,(\eta F' - 3F) = 0 \tag{9a}$$

The above Eqns. (8) and (9a) take the simplest form when A is taken equal to β and B is equal to β^2.

We are essentially left with the third order ordinary differential equation

$$3FF''' - F'F'' + (\eta F' - 3F) = 0 \tag{9b}$$

which requires three boundary conditions to solve.

By analogy with usual boundary-layer formulation, we take the conditions at the edge close to the equator to be

$$\eta \to 0 : u \to u_0\,(x)\,, \quad v \to v_0\,(x) \tag{10}$$

One possible condition for large η is such that the non-linear terms become less significant. The examination of (9) immediately suggests that in the case $(\eta F' - 3F)$ must approach zero at large η. But this would require that $F \sim \eta^3$ for $\eta \to \infty$ and consequently $u \sim \eta^2$, which is a rather restrictive condition on the similarity solution.

Now let us examine the behaviour near the equator by assuming that the solution can be represented by the convergent power series

$$F(\eta) = \sum_{n=0} a_n\,\eta^n, \quad 0 \leqslant \eta \leqslant \eta_0. \tag{11}$$

The equatorward boundary condition then becomes,

$$\eta \to 0 : u = Ag^2 [a_1 + 2a_2\eta + 3a_3\eta^2 + \ \ldots] \to Aa_1g^2 ;$$

$$v = Ag^2 g' [-3a_0 - 2a_1\eta - a_2\eta^2 + \ \ldots] \to -3\ Aa_0g^2g' \qquad (12)$$

An interesting possibility is that as we approach the equator, $u \to 0$. Typical mean monthly flow pattern at 1 km for July (Findlater 1971) for the western part of Indian Ocean near the equator (Fig. 1) suggests that this possibility is of considerable interest very close to the equator. It might be expected that the approximation (3) in such an event might not hold all the way to the equator. Indeed, this case requires to be a_1 is zero so that fu behaves like η^2, while the term vv_y, which is neglected in the meridional balance, behaves like η. This particular case can be handled by a sublayer located at the equator, where a supplementary approximation is required*.

4. Equatorial sublayer

We now construct a supplementary approximation of a sub-region near equator whose meridional length scale is smaller than the previous region. Further, we focus our attention on the case when the interhemispheric transport is significant. In view of the flow patterns near the equator in July (Findlater 1971), we take U/V to be 0 in (1) in the subregion or sublayer. Let U/V be $\bar{\varepsilon}$. L is still comparable to the earlier case, and the meridional scale $\bar{l}$ is smaller than the scale in the previous case. Let $\bar{l}/L$ be $\bar{\delta}$. The orders of the terms in (1) are found by taking ϕ to be of the order V^2 as

$$\bar{\varepsilon}^2 (V^2/L),\ (\bar{\varepsilon}/\bar{\delta})\ (V^2/L),\ (\bar{\delta}/\bar{R}_0)\ (V^2/L),\ (V^2/L)\ ;$$

$$\bar{\varepsilon} (V^2/L),\ (1/\bar{\delta})\ (V^2/L),\ (\bar{\varepsilon}\bar{\delta}/\bar{R}_0)\ (V^2/L),\ (1/\bar{\delta})\ V^2/L\ ;$$

$$\bar{\varepsilon}\,(V/L),\ (1/\bar{\delta})\ (V/L)\ ,$$

where $\bar{R}_0$ is the Rossby number $V/\beta L^2$. Clearly, the continuity equation cannot retain both the terms. The approximation which allows both nonlinear inertial terms as well as coriolis force terms require that $\bar{R}_0$ to be of order $\bar{\delta}$, and $\bar{\varepsilon}$ also to be of order $\bar{\delta}$. If we take L to be of order 3.3×10^3 km for consistency with earlier estimates, and l as, say, 100 km, $\bar{\varepsilon}$ would be 0.03. On the other hand with V equal to 15 m sec^{-1}, and $\beta = 2.3 \times 10^{-11}$m^{-1} sec^{-1}, $\bar{R}_0$ turns out to be 0.059. So the requirement of $\bar{R}_0$ and $\bar{\varepsilon}$ of being of the same order is plausible very close to the equator under these conditions. The resulting equations are

$$vu_y - \beta yv = -\phi_x,$$

$$vv_y = -\phi_y,$$

$$v_y = 0. \qquad (14)$$

The degenerate form of continuity equation leads to the conclusion that v and ϕ depend only on x. To put it differently, the variation of meridional velocity and geopotential in the meridional direction is not significant in this sublayer. The first equation can then be integrated to give

$$u = \tfrac{1}{2}\beta y^2 - (\phi_x/v)\ y + u\,(x{,}0)\ . \qquad (15)$$

5. Matching of the equatorial sublayer

We examine the possibility of matching the sublayer with the region dealt with earlier. At large y, the u component behaves like

$$u \sim \tfrac{1}{2}\ \beta y^2 \qquad (16)$$

On the other hand, the similarity solution given by

$$u \sim Ag^2 [a_1 + 2a_2\eta + 3a_3\eta^2 + \ \ldots\ldots]$$

Then matching requires that a_1 and a_2 are zero. Also,

$$u \sim Ag^2.\ 3a_3\,(y/g)^2\ = 3A\ a_3y^2$$

Since A was chosen to be β, matching requires that a_3 be 1/6. Also, it is clear that the boundary

*If both u_0 and v_0 (10a) are zero, the above expansion may hold upto the equator. In this case, a_0 and a_1 are zero. Suppose a_2 is not, then uv_x, vv_y behave like η^3, while fu behaves like η^2.

condition v_0 on the normal velocity in (10) is simply the velocity component $v(x)$ given by (10).

6. Concluding remarks

Preceding arguments indicate the possibilities of constructing approximations in the equatorial region which give special weightage to the rapid variation of coriolis factor f. As the length scale in meridional direction is smaller than that in the zonal direction, some special features appear. In particular, both coriolis and non-linear inertia terms are present in zonal momentum balance. Also, the arguments indicate that we may visualise two regions. In the low latitudes somewhat away from the equator, the zonal flow dominates. However, if there is considerable interhemispheric transport the vicinity of the equator behaves differently and we need to consider a subregion very close to the equator where the meridional flow dominates.

The main approximation given in section 2 retains the non-linear inertial terms only in the zonal balance, while the meridional balance is geostrophic. Also, arguments are given to show that the case of special interest for studies of monsoon circulation requires a supplementary approximation for an equatorial sublayer. Arguments are given to show that the approximation in the sublayer can be matched with that in the main region.

The above preliminary results indicate possibilities of approximations which retain nonlinearity in the lowest order and which are designed for equatorial and which could be extended to tropical regions.

Acknowledgement

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DISCUSSION

John A. Young : I wish to compliment the author for his contribution towards establishing a framework for studying mean cross-equatorial problem and I believe that the results will be most significant if longitudinal variations can be included. Does your theory allow for such variations on a relatively small scale ?

Author : Thank you for your comments. The present approximation deals with the case in which the zonal length scale is larger than the meridional scale. There are possibilities of constructing approximations for features like Somali jet where the zonal length scale might be smaller than the meridional scale.