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On the reliability of iterative solutions for Poisson's type equation

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ABSTRACT. The accuracy attainable using the Gauss Seidel iterative method for Poisson's equation is investigated for different number of iterations and for different number of mesh points in a rectangualr area.

It is shown that in general iterative methods of solution of Poisson's type of equation are not to be recommended for atmospheric problems.

1. Introduction

In numerical computation of atmospheric motion we often come across equations of form

$$\nabla^2 \psi = \sigma \tag{1}$$

where for example σ could be the vorticity whose spatial distribution is known and ψ the unknown could be the stream function.

There are various methods of soving this equation. Owing to their simplicity iterative methods of solution have been popular amongst meteorologists.

The purpose of this paper is to point out some short-comings of iterative schemes.

2. Iterative solution of Poisson's equation

The five point finite difference form of Eqn. (1) in rectangular coordinates (X, Y) is:

$$\psi_{x,y} = \tfrac{1}{4}(\psi_x \psi_{y+H} + \psi_{x,y-H} + \psi_{x+H,y} + \\ + \psi_{x-H,y} - H^2 \sigma_{x,y}) \tag{2}$$

where H is the grid size.

There are various iterative schemes possible. We have the Jacobi scheme which calculates a set of iterates exclusively in terms of the previous set. This is hardly used because there are other schemes which are not only faster but also demand less storage. An example of this is the Gauss Seidel iteration which involves the use of recent iterate as they become available. Details of these two schemes can be obtained in many text-books on numerical methods, example of which is Smith (1971).

The Young (1954) successive over relaxation SOR, the Alternating Direction Implicit Scheme ADI, proposed by Peaceman and Rachford (1955) and subsequent modifications of these are the fastest iterative schemes presently available. These fast schemes reduce the time to obtain a given accuracy by a factor which is critically dependent on the accelerating factor. For example with the optimum accelerating factor SOR is about 40 times faster than Gauss Seidel. When this accelerating factor is altered by about one per cent, the scheme is only 20 times faster (*see* Smith 1971, p. 150).

Thus it is safe to say that SOR is only a few times faster than Gauss Seidel. This statement is especially true in the case of other elliptic equations for which it is difficult to calculate the accelerating factor. We shall therefore limit our discussion to Gauss Seidel.

2. Numerical test solutions of Poisson's equation

In order to test the reliability of iterative solutions, random numbers were generated and assigned to values of ψ at the grid points of a rectangular area. The finite difference form of the Laplacian σ as given in equation (2) were than calculated at each of the grid points. Thus the set of ψ, which henceforth we shall denote by T represents the true solution of the given set of σ.

Now taking the set of σ as the known quantities from which the ψ were to be calculated, the Gauss Seidel technique was applied on Eqn. (2) a number of times to obtain better and better values for ψ. At each iteration we calculated the successive fractional error E which for the kth iteration is defined as

$$E_k = \Sigma \mid T - \psi_k \mid / \Sigma \mid T \mid$$

where summation is over all the grid points except at the boundaries where values of ψ were known and fixed.

Table 1 illustrates the successive accuracy attained for defferent number of mesh points.

TABLE 1

No. of itera-tions	No. of grid meshes 21×21	21×41	21×71
0	$1\cdot 00\times 10^{0}$	$1\cdot 00\times 10^{0}$	$1\cdot 00\times 10^{0}$
100	$1\cdot 01\times 10^{-1}$	$2\cdot 39\times 10^{-1}$	$2\cdot 91\times 10^{-1}$
200	$8\cdot 50\times 10^{-3}$	$5\cdot 18\times 10^{-2}$	$7\cdot 65\times 10^{-2}$
300	$7\cdot 13\times 10^{-4}$	$1\cdot 11\times 10^{-2}$	$2\cdot 02\times 10^{-2}$
400	$6\cdot 00\times 10^{-5}$	$2\cdot 38\times 10^{-3}$	$5\cdot 36\times 10^{-2}$
500	$5\cdot 03\times 10^{-6}$	$5\cdot 09\times 10^{-4}$	$1\cdot 42\times 10^{-3}$
600	$4\cdot 22\times 10^{-7}$	$1\cdot 09\times 10^{-4}$	$3\cdot 74\times 10^{-4}$
700	$3\cdot$[illegible]4×10^{-8}	$2\cdot 31\times 10^{-5}$	$9\cdot 88\times 10^{-5}$
800	$2\cdot 97\times 10^{-9}$	$4\cdot 94\times 10^{-6}$	$2\cdot 61\times 10^{-5}$
900	$2\cdot 49\times 10^{-10}$	$1\cdot 05\times 10^{-6}$	$6\cdot 86\times 10^{-6}$
1000	$2\cdot 10\times 10^{-11}$	$2\cdot 24\times 10^{-7}$	$1\cdot 81\times 10^{-6}$

3. Results and conclusion

We note that change in ψ at a particular grid point will have effect on the ψ at adjacent grid points after 1 iteration. Thus we can see that this change will be noticed at grid point, n grids away after n iterations. Thus we can conclude as shown in Table 1 that the larger the number of grid points in the rectangle, the more the number of iterations for a modification to propagate through the whole system of grids; and hence the less the rapidity of convergence of the iterative scheme. After a certain number of grid points the convergence hardly changes with increase in number of grid points because the mutual effects between the grid points become insignificant when the two points are separated by a large number of grid points.

In atmospheric motion we can contemplate a network of 21×71 points as adequate. We can assume an accuracy of about 1 in 10 in the meteorological parameter by the process of time extrapolation and this will serve as an initial guess.

Thus from the table we see that to improve the accuracy from about 1 in 10 to 1 in 10^4 we shall have to do over 600 iterations. This is extremely time consuming on the computer. And even then this accuracy may not be good enough especially if the calculated ψ have to be differentiated in some other part of the problem. The process of differentiating involves subtracting two large quantities to obtain a smaller one and so the accuracy reduces considerably.

It is thus clear that one must be cautioned in placing reliability on problems which involve Poisson's equation solved by iterative technique—otherwise the calculated wave may be merely the accumulated errors caused by the inexactitude of the iterative scheme.

We may also note that other iterative schemes merely reduce the required number of iterations by a small factor ; and consequently irrespective of the particular scheme, the number of iterations required to obtain a tolerable accuracy is still prohibitive.

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