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PROPAGATION OF WAVES ON THE PLANETARY SURFACES

Most of the problems arising in meteorology and related areas get local treatment, *i.e.*, one considers them on a small patch of the planetary surface. The small patch can be dealt with as a part of $\mathbf{R}^2$ and thus one can write down partial differential equations describing certain processes in terms of the Euclidean coordinates.

Dealing with global set up of the same problems are much more difficult, mainly because of the non-zero curvature of the planetary surfaces.

Propagation of certain types of wave on the surfaces can be described by the invariant wave equation which is generalization of the classical wave equation. If we denote by ξ^1, ξ^2 local coordinates and by

$$dS^2 = g_{ij}\, d\xi^i d\xi^j$$

the metric form on the surface then the invariant wave operator (Friedlander 1976) is given by the formula :

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \sqrt{g}\, g^{ij} \frac{\partial}{\partial \xi^j} + \frac{\rho(\xi)}{8} \qquad (1)$$

where $\rho(\xi)$ is the scalar curvature at point ξ.

The initial value problem for the wave operator consists of finding solutions of

$$\Box u = 0$$

which at time $t=0$ satisfy the initial conditions

$$u(0, \xi) = f(\xi)$$

$$\frac{\partial u\,(0, \xi)}{\partial t} = g(\xi)$$

In the first approximation we can think of the planetary surfaces as spheres with longitude λ and co-latitude θ serving as local coordinates. In this case the metric form and the invariant wave operator correspondingly are

$$dS^2 = R^2\,(d\theta^2 + \sin^2\theta\; d\lambda^2)$$

and $$\Box = \frac{R^2}{v^2}\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial\theta^2} + \text{cotan}\,\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\right) + \frac{1}{4} \qquad (2)$$

where, R is the planetary radius and v is the speed of propagating waves.

The initial value problem on the sphere consists of finding the functions $u(t, \theta, \lambda)$ such that

$$\begin{aligned} \Box u\,(t, \theta, \lambda) &= 0 \\ u(0, \theta, \lambda) &= f(\theta, \lambda) \\ \frac{\partial u\,(0, \theta, \lambda)}{\partial t} &= \frac{v}{R}\, g\,(\theta, \lambda). \end{aligned} \qquad (3)$$

It is well known that the solution of Eqn. (3) can be written as a series in terms of spherical harmonics (John 1978). We show how to obtain the solution of Eqn. (3) in explicit form.

To simplify our work we temporarily assume that $f=0$ and $R=v=1$. The general case follows from this particular case as it is shown in John (1978) and the general solution will be written up later on in Eqn. (7).

Rather than using heavy mathematical computations, we use phenomenological approach. Recall that on small areas planetary surface is flat. That is, if globally u satisfies Eqn. (3) then locally it should behave like a solution of the wave equation on the plane. But the solution of the wave equation on the plane is given by the formula (John 1978) :

$$u\,(P, t) = \frac{1}{2\pi}\int_{d<t} \frac{g\,(\xi)\, d\mu\,(\xi)}{\sqrt{t^2 - d^2\,(P, \xi)}} \qquad (4)$$

where, $u(P, t)$ is the value of u at point P at time t, ξ are the local coordinates of a point on the sphere, $d\mu\,(\xi)$ is the Haar measure (*i.e.*, measure invariant with respect to natural motions) and d is the distance between P and ξ.

We can then look for a solution of Eqn. (3) in the form :

$$u\,(P, t) = \frac{1}{2\pi}\int_{D(P,t)} \frac{g\,(\xi)\, d\mu\,(\xi)}{\sqrt{|h(t) - h(d)|}} \qquad (5)$$

with ξ, $d\mu(\xi)$, d, $D\,(P, t)$ being correspondingly local coordinates, Haar measure, distance between P and ξ, the disc $d \leqslant t$ on the sphere and h is an unknown function that will be determined later on. One can easily compute that in terms of coordinates θ, λ.

$$d\mu = \sin\theta\; d\theta\; d\lambda$$

and thus

$$u\,(\theta, \lambda, t) = \frac{1}{2\pi}\int_{D(\theta,\lambda,t)} \frac{g(\psi, \varphi)\sin\psi\;\, d\psi\, d\varphi}{\sqrt{|h\,(t) - h\,(d)|}} \qquad (6)$$

where, θ, λ and ψ, φ are correspondingly coordinates of P and ξ.

Substituting Eqn. (6) into Eqn. (3) and using that on the sphere the distance between points (θ, λ) and (ψ, φ) satisfies :

$$\cos d = \cos\theta\cos\psi + \sin\theta\sin\psi\cos(\lambda - \varphi)$$

we obtain

$$h'' + h = 0$$

that gives us

$$h(t) = \sqrt{2}\cos t.$$

This leads us to the conclusion that

$$u(\theta, \lambda, t) = \frac{1}{\sqrt{8}\,\pi}\int_{D(\theta,\lambda,t)} \frac{g\,(\psi, \varphi)\,\sin\psi\; d\psi\; d\varphi}{\sqrt{\cos\theta\cos\psi + \sin\theta\sin\psi\cos(\lambda - \varphi) - \cos t}}$$

One can verify the validity of the latter by substituting it into Eqn. (3).

For completeness sake we give here a formula for the general solution of Eqn. (3) which is :

$$\begin{aligned} u(\theta, \lambda, t) = {} & \frac{R}{\sqrt{8}\,\pi v}\int_{D(\theta,\,\lambda,\,\frac{vt}{R})} \frac{g\,(\psi, \varphi)\sin\psi\, d\psi\, d\varphi}{\sqrt{\cos\theta\cos\psi + \sin\theta\sin\psi\cos(\lambda - \varphi) - \cos t}} \\ & + \frac{R}{\sqrt{8}\,\pi v}\frac{\partial}{\partial t}\int_{D(\theta,\lambda,\,\frac{vt}{R})} \frac{f(\psi, \varphi)\sin\psi\; d\psi\; d\varphi}{\sqrt{\cos\theta\cos\psi + \sin\theta\sin\psi\cos\,(\lambda - \varphi) - \cos t}} \end{aligned} \qquad (7)$$

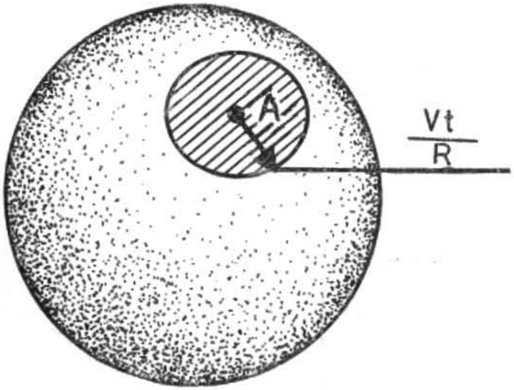

Fig. 1

where $D(\theta, \lambda, \frac{vt}{R})$ is the geodesic disc of radius $\frac{vt}{R}$ centred at the point (θ, λ), *i.e.*, the set of all points on the sphere whose geodesic distance from the point (θ, λ) is less than $\frac{vt}{R}$.

The formula Eqn. (7) holds for $0 \leqslant \frac{vt}{R} \leqslant \pi$, when $\frac{vt}{R} = \pi$, $D(\theta, \lambda, \frac{vt}{R}) = S^2$.

One can use the above formula to compute $u(t, \theta, \lambda)$ and $\frac{\partial u(t, \theta, \lambda)}{\partial t}$ for $\frac{vt}{R} \in [0, \pi]$ and then solve it again for $\frac{vt}{R} \in [\pi, 2\pi]$ etc.

To understand behaviour of the solutions better we look at the point $\theta = 0$. The formula for u simplifies to the form :

$$u(t,0,\lambda) = \frac{R}{\sqrt{8\pi v}} \int_{D(0,\frac{vt}{R})} \frac{g(\psi,\varphi) \sin\psi \, d\psi \, d\varphi}{\sqrt{\cos\psi - \cos t}} + \frac{I}{\sqrt{8\pi v}} \frac{\partial}{\partial t} \int_{D(0,\frac{vt}{R})} \frac{f(\psi,\varphi) \sin\psi \, d\psi \, d\varphi}{\sqrt{\cos\psi - \cos t}}.$$

The value of u at the point $\theta = 0$ are affected only by the values of the initial data on the disc $D(0, \frac{vt}{R})$. The initial data near the boundary of the disc provides more influence than the initial data inside the disc because of presence of the factor $\frac{\sin\psi}{\sqrt{\cos\psi - \cos t}}$. In general the value of the solution at a point A depends on the initial data on the geodesic disc of radius $\frac{vt}{R}$ (shaded in Fig. 1). As time goes on the disc increases until it covers the whole surface. At that momemt $(t = \frac{R\pi}{v})$ the formula Eqn. (7) is no longer valid; we have to compute the data at $t = \frac{R\pi}{v}$ and then use formula Eqn. (7) with these data to compute the solution for $\frac{vt}{R} \in [\pi, 2\pi]$ etc.

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