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Dispersion of *SH* waves in a laterally and radially inhomogeneous layered earth model

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ABSTRACT. Love waves in a spherical Earth Model, wherein μ and ρ are varying laterally and radially, have been discussed in this paper. The ratio μ/ρ is taken to be independent of r and θ. Thomson-Haskell method has been used to discuss the wave propagation in mutli-layered model. Dispersion equation for a layer over a rigid shell, two layered model, three layered model with liquid core and three layered model with solid core have been obtained.

1. Introduction

Dispersion properties of seismic waves are widely used to study the detailed structure of the earth's interior. The dispersion of surface waves depends upon the elastic properties of the medium through which the waves propagate. Considerable amount of work has been done on the propagation of elastic waves in an earth's model assumed to be vertically inhomogeneous. It is now well established that the density and elastic properties of the earth vary both vertically and laterally.

Love wave propagation in a vertically and laterally inhomogeneous medium has been studied by some atuhors. Bhattacharya (1970b) studied the love wave dispersion in a laterally heterogeneous layer lying over a homogeneous semi-infinite medium by assuming that, in the layer,

$$\beta = \beta_0\,(1+bx), \quad \mu = \mu_0\,(1+mx)$$

He obtained the dispersion curves for the first two modes. Chatterjee (1972) discussed the problem of dispersion of love waves in a laterally and in a vertically heterogeneous layer lying over a homogeneous half space. He assumed that, in the layer

$$\beta = \beta_0\, e^{px}, \quad \mu = \mu_0\, e^{qx}$$

He also obtained the love wave dispersion curve for first two modes.

Singh *et al.* (1976) discussed the propagation of love waves in laterally and vertically inhomogeneous layered model. They assumed that μ and ρ are both function of x and z and μ/ρ is independent of x. They used the Thomson-Haskell matrix method (Thomson 1950, Haskell 1953) to obtain the dispersion equation. They have obtained the dispersion equations in explicit form for the half-space, for a layer over a rigid bottom and for a two layered half-space. The above authors have assumed the earth model to be a layered half-space. Bhattacharya (1976) assumed the earth to be a layered spherical model and the Thomson-Haskell matrix method is applied to non-homogeneous spherical layered model. He assumed a source in one of the layers and calculate the displacements at the surface.

We have considered the earth model to be layered spherical one and have discussed the propagation of love waves in such a structure, assuming μ and ρ to be functions of both r and θ and μ/ρ independent of r and θ. Thomson-Haskell method has been used to obtain the dispersion equation in the layered structure. Dispersion equations have been obtained for a layer over a rigid sphere, three layered model with all the layers solid and a three layered model when the inner sphere is assumed to be liquid.

2. Equation of motion and its solution

We consider the earth to be a layered spherical model and use (r, θ, ϕ) as spherical polar co-ordinates with origin at the centre of the earth. We are working on the problem of the decomposition of the vector wave equation in the spherical polar co-ordinates. The scalar wave equation of *SH*-wave is obtained to be

$$\mu\left(\frac{\partial^2\omega}{\partial r^2}+\frac{2}{r}\frac{\partial\omega}{\partial r}-\frac{2\omega}{r^2}\right)+\frac{\partial\mu}{\partial r}\left(\frac{\partial\omega}{\partial r}-\frac{\omega}{r}\right)$$
$$+\frac{\mu}{r^2}\left[\frac{\partial^2\omega}{\partial\theta^2}+\frac{\partial\omega}{\partial\theta}\cot\theta+\omega\left(1-\cot^2\theta\right)\right.$$
$$\left.+\frac{1}{\mu}\frac{\partial\mu}{\partial\theta}\left(\frac{\partial\omega}{\partial\theta}-\omega\cot\theta\right)\right]=\rho\,\frac{\partial^2\omega}{\partial t^2} \qquad (1)$$

where ω, the displacement component and μ and ρ, the elastic parameters, are functions of r and θ only.

Let us assume that

$$\omega = R(r)\ \Theta(\theta)\ e^{i\omega t} \tag{1a}$$
$$\mu = \mu_0\, p(r)\ q(\theta) \tag{1b}$$
$$\rho = \rho_0\, p(r)\ q(\theta) \tag{1c}$$

By the method of separation of variables, we obtain

$$\frac{d^2R}{dr^2} + \frac{dR}{dr}\left[\frac{2}{r} + \frac{1}{p}\frac{dp}{dr}\right] + \left[\frac{\omega^2}{\beta_0^2} - \frac{1}{r}\left(\frac{1}{p}\frac{dp}{dr} + \frac{2}{r}\right) - \frac{N}{r^2}\right] R = 0 \tag{2a}$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{d\Theta}{d\theta}\left[\cot\theta + \frac{1}{q}\frac{dq}{d\theta}\right] + \left[1 + N - \cot\theta\left(\cot\theta + \frac{1}{q}\frac{dq}{d\theta}\right)\right]\Theta = 0 \tag{2b}$$

where N is separation constant and $\beta_0^2 = \mu_0/\rho_0$

we assume that $\cot\theta + \frac{1}{q}\frac{dq}{d\theta} = 0$ (3)

This gives us $q = \frac{a}{\sin\theta} \qquad 0 \leqslant \theta \leqslant \pi$

$$= \frac{-a}{\sin\theta} \qquad \pi < \theta \leqslant 2\pi,\ a > 0 \tag{4}$$

where a is a constant.

It is evident from Eqn. (4) that q is ∞ at $\theta = 0$ and $\theta = \pi$ the rigidity becomes ∞ on the line passing through the poles. With the above assumption, the Eqn. 2(b) becomes

$$\frac{d^2\Theta}{d\theta^2} + k^2\theta = 0 \tag{5}$$

where, $k^2 = N + 1$

since $k^2 > 0$, therefore $N > -1$ (6)

The solution of Eqn. (5) is

$$\Theta = e^{\pm ik\theta}$$

We take the solution with negative sign and thus from the Eqn. (1a), we obtain

$$\omega = Re^{i(\omega t - k\theta)} \tag{7}$$

to be a solution of Eqn. (1).

We assume that $p(r)$ is such that

$$\frac{2}{r} + \frac{1}{p}\frac{dp}{dr} = \frac{b}{r}$$

where b is some constant.

This gives

$$p = cr^{b-2} \tag{8}$$

where c is an arbitrary constant.

Eqn. (2a) takes the form

$$r^2\frac{d^2R}{dr^2} + rb\frac{dR}{dr} + \left[\frac{\omega^2}{\beta_0^2}r^2 - (b + k^2 - 1)\right] R = 0 \tag{9}$$

Let us assume that

$$R = r^m u(Z),\ Z = r\frac{\omega}{\beta_0},\ 2m = 1 - b,$$

thus the Eqn. (9) reduces to

$$Z^2\frac{d^2u}{dZ^2} + Z\frac{du}{dZ} - (P^2 - Z^2)U = 0 \tag{10a}$$

where $P^2 = (1+b)^2 + 4(K^2 - 1)$ (10b)

Eqn. (10) is the Bessel's equation of order ρ and its solution is given by

$$U = A\, J_P(Z) + B\, J_{-P}(Z), \tag{11}$$

where $J_P(Z), J_{-P}(Z)$ are Bessel's functions of first kind and A and B are constants.

The solution of Eqn. (9) is given by

$$R = r^m\left[AJ_P\left(r\frac{\omega}{\beta_0}\right) + BJ_{-P}\left(r\frac{\omega}{\beta_0}\right)\right] \tag{12}$$

Thus the solution of Eqn. (1) is given by

$$\omega = r^m\left[A\,J_P\left(r\frac{\omega}{\beta_0}\right) + BJ_{-P}\left(r\frac{\omega}{\beta_0}\right)\right] e^{i(\omega t - k\theta)} \tag{13}$$

3. n-layered model and frequency equation

We consider the earth to be a solid sphere made up of n concentric shells with the nth one to be a sphere of radius R_0 (say). The lateral variation in each shell is assumed to be the same whereas the radial variation is different. We shall use subscript M to denote the parameters of the Mth layer. The layers are numbered as shown in the Fig. 1. The Mth layer is bounded by $r = r_{M-1}$ at the top and by $r = r_M$ at the bottom. Where r_M means value of r at the Mth interface. The elastic parameters μ and ρ in the Mth layer are given by

$$\mu_M(r,\theta) = \mu_{0M}\, p_M(r)\ q(\theta), \tag{14}$$
$$\rho_M(r,\theta) = \rho_{0M}\, p_M(r) q(\theta)$$

where $p_M(r) = C_M\, r^{b_M - 2}$

μ_{0M} and ρ_{0M} are constants.

The sheer wave velocity β_M, in the Mth layer, is given by

$$\beta_M = \left(\frac{\mu_M}{\rho_M}\right)^{\frac{1}{2}} = \left(\frac{\mu_{0M}}{\rho_{0M}}\right)^{\frac{1}{2}} = \beta_{0M} = \text{constant} \tag{15}$$

The displacement component in the Mth layer is given by

$$\omega_M = W_M(r)\, e^{i\,(\omega t - k\theta)} \tag{16}$$

where,

$$W_M(r) = r^{m^{(M)}} \left[A_M J_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) + B_M J_{-P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) \right] \tag{17}$$

and

$$m^{(M)} = \frac{1 - b_M}{2},$$

$$P_M^2 = (1 + b_M)^2 + \mu\,(K^2 - 1) \tag{18}$$

Here onward we shall write $m^{(M)}$ for m in the Mth layer.

The shear stress $T_{r\phi}$ in the Mth layer is given by

$$[Tr\phi]_M = T_M\,(r) q(\theta) e^{i(\omega t - K\theta)} \tag{19}$$

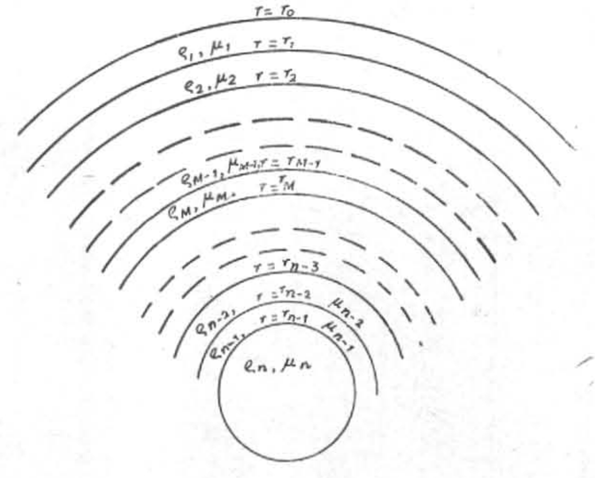

Fig 1. *n* layered spherical earth model

where,

$$T_M(r) = \mu_{0M} p_M(r) \left\{ A_M \left[r^{m^{(M)}} J^1_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) + \left(m^{(M)} - 1 \right) r^{m^{(M)}-1} J_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) \right] + B_M \left[r^{m^{(M)}} J^1_{-P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) + \left(m^{(M)} - 1 \right) r^{m^{(M)}-1} J_{-P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) \right] \right\}. \tag{20}$$

Eqns. (17) and (20) can be written in the matrix form as

$$\begin{bmatrix} W_M(r) \\ T_M(r) \end{bmatrix} = D_M(r) \begin{bmatrix} A_M \\ B_M \end{bmatrix}, \qquad r_{M-1} \leqslant r \leqslant r_M \tag{21}$$

where elements of $D_M(r)$ given by

$$D_{11} = r^{m^{(M)}} J_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right), \qquad D_{12} = r^{m^{(M)}} J_{-P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right),$$

$$D_{21} = \mu_{0M} p_M(r) \left[r^{m^{(M)}} J^1_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) + r^{m^{(M)}-1} \left(m^{(M)} - 1 \right) J_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) \right],$$

$$D_{22} = \mu_{0M} p_M(r) \left[J^1_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) + r^{-1} \left(m^{(M)} - 1 \right) J_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) \right] r^{m^{(M)}}.$$

The inverse of the marix $D_M(r)$, *i.e.*, $D_M^{-1}(r)$ is given by

$$D_M^{-1}(r) = \frac{}{\mu_{0M} p_M\,(r)^{2m^{(M)}} \left[J_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) J^1_{-P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) - J_{-P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) J^1_{P_M}\left(r\, \frac{\omega}{\beta_{0M}} \right) \right]} \times \begin{bmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{11} \end{bmatrix} \tag{22}$$

From Eqn. (21), we have

$$\begin{bmatrix} A_M \\ B_M \end{bmatrix} = \overset{-1}{D_M}(r) \begin{bmatrix} W_M(r) \\ T_M(r) \end{bmatrix}, \ r_{M-1} \leqslant r \leqslant r_M \quad (23)$$

Eqns. (21) and (23) give

$$\begin{bmatrix} W_M(r_M) \\ T_M(r_M) \end{bmatrix} = G_M \begin{bmatrix} W_M(r_{M-1}) \\ T_M(r_{M-1}) \end{bmatrix} \quad (24)$$

where,

$$G_M = D_M(r_M) \overset{-1}{D_M}(r_M - 1) \quad (25)$$

Since the displacement ω and the shear stress $T_r\phi$ are continuous across each interface, therefore, from the Eqns. (17) and (19) we conclude that the continuity of ω and $T_{r\phi}$ at each interface depends upon the continuity of the expressions for $W_M(r)$ and $T_M(r)$ at each interface, thus

$$W_M(r_{M-1}) = W_{M-1}(r_{M-1}),$$
$$T_M(r_{M-1}) = T_{M-1}(r_{M-1}) \quad (26)$$

with the help of Eqn. (26), Eqn. (24) takes the form

$$\begin{bmatrix} W_M(r_M) \\ T_M(r_M) \end{bmatrix} = G_M \begin{bmatrix} W_{M-1}(r_{M-1}) \\ T_{M-1}(r_{M-1}) \end{bmatrix} \quad (27)$$

Using Eqn. (27) respectively, alongwith the Eqns. (24) and (26), we finally obtain

$$\begin{bmatrix} W_n(r_{n-1}) \\ T_n(r_{n-1}) \end{bmatrix} = J \begin{bmatrix} W_1(r_0) \\ T_1(r_0) \end{bmatrix} \quad (28)$$

where r_0 is the radius of the sphere and $r_{n-1} = R_0$ is the radius of the inner sphere as shown in the Fig. 1. and

$$J = G_{n-1} \ G_{n-2} \ \ldots\ldots \ G_2 \ G_1 \quad (29)$$

Since the surface of the sphere is stress free, therefore,

$$T_1(r_o) = 0$$

This gives

$$\frac{W_n(R_0)}{T_n(R_0)} = \frac{J_{11}}{J_{21}} \quad (30)$$

Now the displacement component in the nth layer can be written as :

$$W_n(r) = r^{m^{(n)}} \left[A_n J_{P_n}\left(r \frac{\omega}{\beta_{0n}} \right) + \right.$$
$$\left. + B_n J_{-P_n}\left(r \frac{\omega}{\beta_{0n}} \right) \right],$$

From radiation condition, we have $B_n = 0$ therefore

$$W_n(r) = A_n r^{m^{(n)}} J_{P_n}\left(r \frac{\omega}{\beta_{0n}} \right),$$

where, $P_n + m^{(n)} > 0$

and $$W_n(R_0) = A_n R_0^{m^{(n)}} J_{P_n}\left(R_0 \frac{\omega}{\beta_{0n}} \right),$$

Thus we obtain

$$T_n(R_0) = \mu_{0n} p_n(R_0) \ A_n \left[R_0^{m^{(n)}} J^1_{P_n}\left(R_0 \frac{\omega}{\beta_{0n}} \right) \right.$$
$$\left. + \left(m^{(n)} - 1 \right) R_0^{(m^{(n)} - 1)} \ J_{P_n}\left(R_0 \frac{\omega}{\beta_{0n}} \right) \right] \quad (31)$$

The Eqn. (30) now takes the form

$$J_{21} R_0^{m^{(n)}} J_{P_n}\left(R_0 \frac{\omega}{\beta_{0n}} \right) =$$
$$J_{11} \mu_{0n} p_n(R_0) \left[R_0^{M^{(n)}} J^1_{P_n}\left(R_0 \frac{\omega}{\beta_{0n}} \right) + \right.$$
$$\left. \left(m^{(n)} - 1 \right) R_0^{m^{(n)} - 1} J_{P_n}\left(R_0 \frac{\omega}{\beta_{0n}} \right) \right] \quad (32)$$

This is the required frequency equation for the n layered model

4. Two layered model

The dispersion equation for two layered model can be obtained from the Eqn. (32) by putting $n=2$. Thus we obtain

$$J_{21} R_0^{m^{(2)}} J_{P_2}\left(R_0 \frac{\omega}{\beta_{02}} \right) =$$
$$J_{11} \mu_{02} p_2(R_0) \left[J^1_{P_2}\left(R \frac{\omega}{\beta_{02}} \right) + \right.$$
$$\left. \left(m^{(2)} - 1 \right) R_0^{m^{(2)} - 1} J_{P_2}\left(R_0 \frac{\omega}{\beta_{02}} \right) \right] \quad (33)$$

where the elements of the matrix

$$J = G_1 = D_1(r_1) D_1^{-1}(r_0)$$

are given by

$$J_{11} = r_1^{m^{(1)}} \left\{ J_{P_1}\left(R_1 \frac{\omega}{\beta_{01}} \right) \left[r_0 J^1_{-P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) \right. \right.$$
$$\left. + \left(m^{(1)} - 1 \right) J_{-P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) \right] -$$
$$- J_{-P_1}\left(r_1 \frac{\omega}{\beta_{01}} \right) \left[r_0 J^1_{P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) + \right.$$
$$\left. \left. \left(m^{(1)} - 1 \right) J_{P_1}\left(r_0 \frac{\omega}{\beta_1} \right) \right] \right\} \Big/$$
$$r_0^{m^{(1)}+1} \left[J_{P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) J^1_{-P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) \right.$$
$$\left. - J_{-P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) J^1_{P_1}\left(r_0 \frac{\omega}{\beta_{01}} \right) \right]$$

$$J_{21} = r_0^{-(\overset{(1)}{m}+1)} \, r_1^{\overset{(1)}{m}-1} \, \mu_{01}\, p_1(r_1) \left[J_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) J^1_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) - J_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) J^1_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right]^{-1}$$

$$\left\{ \left[r_1 J^1_{P1}\left(r_1 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{P1}\left(r_1 \frac{\omega}{\beta_{01}}\right) \right] \left[r_0 J^1_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] \right.$$

$$\left. - \left[r_1 J^1_{-P1}\left(r_1 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{-P1}\left(r_1 \frac{\omega}{\beta_{01}}\right) \right] \left[r_0 J^1_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] \right\}$$

5. Inhomogeneous layer over a rigid sphere

Let μ_1 and ρ_1 be the rigidity and density respectively in the layer.

Boundary conditions are:

$$\omega = 0 \quad \text{at } r = R_0 , \qquad T_{r\phi} = 0 \quad \text{at } r = r_0 \tag{34}$$

From Eqns. (17), (20) and (34)

$$A_1 J_{P1}\left(R_0 \frac{\omega}{\beta_{01}}\right) + B_1 J_{-P1}\left(R_0 \frac{\omega}{\beta_{01}}\right) = 0 , \tag{35}$$

$$A_1 \left[r_0 J^1_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] + B_1 \left[r_0 J^1_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] = 0 \tag{36}$$

where, $\beta_{01} = \left(\frac{\mu_{01}}{\rho_{01}} \right)^{1/2}$

Eliminating A_1 and B_1 from Eqns. (35) and (36) we get,

$$J_{P1}\left(R_0 \frac{\omega}{\beta_{01}}\right) \left[R_0 J^1_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] - J_{-P1}\left(R_0 \frac{\omega}{\beta_{01}}\right) \left[r_0 J^1_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \right.$$

$$\left. + \left(\overset{(1)}{m}-1\right) J_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] = 0 \tag{37}$$

6. Three layered model (with liquid core)

Here we assume the earth to be a three layered model. The upper two layers are assumed to be solid and the inner one has been taken to be liquid. Such a model has been shown in Fig. 2.

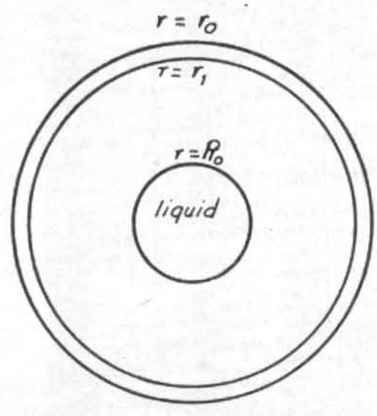

Fig. 2. Three layered model with liquid core

The boundary conditions for such a model are

$$[T_{r\phi}]_{r=r_0} = 0 \tag{38}$$

$$[T_{r\phi}]_1 = [T_{r\phi}]_2, \quad \omega_1 = \omega_2 \qquad \text{at } r = r_1 \tag{39}$$

$$[T_{r\phi}]_2 = 0 \qquad \text{at } r = R_0 \tag{40}$$

Eqns. (38), (39) and (40) give us,

$$A_1 \left[r_0 J^1_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] + B_1 \left[r_0 J^1_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) + \left(\overset{(1)}{m}-1\right) J_{-P1}\left(r_0 \frac{\omega}{\beta_{01}}\right) \right] = 0, \tag{41}$$

$$p_1\left(r_1\right)\mu_{01}\, r_1^{\overset{(1)}{m}-1}\left\{A_1\left[r_1 J^1{}_{P1}\left(r_1\frac{\omega}{\beta_{01}}\right)+\left(\overset{(1)}{m}-1\right)J_{P1}\left(r_1\frac{\omega}{\beta_{01}}\right)\right]+B_1\left[r_1 J^1{}_{-P1}\left(r_1\frac{\omega}{\beta_{01}}\right)+\left(\overset{(1)}{m}-1\right)J_{-P1}\left(r_1\frac{\omega}{\beta_{01}}\right)\right]\right\} = p_2\left(r_2\right)\mu_{02}\, r_1^{\overset{(2)}{m}-1}\left\{A_2\left[r_1 J^1{}_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right)\right]+B_2\left[r_1 J^1{}_{-P2}\left(r_1\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{-P2}\left(r_1\frac{\omega}{\beta_{02}}\right)\right]\right\} \tag{42}$$

$$r_1^{\overset{(1)}{m}}\left[A_1 J_{P1}\left(r_1\frac{\omega}{\beta_{01}}\right)+B_1 J_{-P1}\left(r_1\frac{\omega}{\beta_{01}}\right)\right]=r_1^{\overset{(2)}{m}}\left[A_2 J_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right)+B_2 J_{-P2}\left(r_1\frac{\omega}{\beta_{02}}\right)\right] \tag{43}$$

and finally

$$A_2\left[R_0 J^1{}_{P2}\left(R_0\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{P2}\left(R_0\frac{\omega}{\beta_{02}}\right)\right]+B_2\left[R_0 J^1{}_{-P2}\left(R_0\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{-P2}\left(R_0\frac{\omega}{\beta_{02}}\right)\right]=0\,. \tag{44}$$

Eliminating the constants A_1, B_1, A_2 and B_2 from the Eqns. (41), (42), (43) and (44) we get,

$$|\,A\,| = 0$$

where the elements of the matrix A are given by

$$A_{11}=r_0 J^1{}_{P1}\left(r_0\frac{\omega}{\beta_{01}}\right)+\left(\overset{(1)}{m}-1\right)J_{P1}\left(r_0\frac{\omega}{\beta_{01}}\right),$$

$$A_{12}=r_0 J^1{}_{-P1}\left(r_0\frac{\omega}{\beta_{01}}\right)+\left(\overset{(1)}{m}-1\right)J_{-P1}\left(r_0\frac{\omega}{\beta_{01}}\right),$$

$$A_{13}=0\,,\qquad A_{14}=0\,,$$

$$A_{21}=p_1(r_1)\mu_{01}\left[r_1 J^1{}_{P1}\left(r_1\frac{\omega}{\beta_{01}}\right)+\left(\overset{(1)}{m}-1\right)J_{P1}\left(r_1\frac{\omega}{\beta_{01}}\right)\right]r_1^{\overset{(1)}{m}-1},$$

$$A_{22}=p_1(r_1)\mu_{01}\left[r_1 J^1{}_{-P1}\left(r_1\frac{\omega}{\beta_{01}}\right)+\left(\overset{(1)}{m}-1\right)J_{-P1}\left(r_1\frac{\omega}{\beta_{01}}\right)\right]r_1^{\overset{(1)}{m}-1},$$

$$A_{23}=-\mu_{02}p_2(r_1)\left[r_1 J^1{}_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right)\right]r_1^{\overset{(2)}{m}-1},$$

$$A_{24}=-\mu_{02}p_2(r_1)\left[r_1 J^1{}_{-P2}\left(r_1\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{-P2}\left(r_1\frac{\omega}{\beta_{02}}\right)\right]r_1^{\overset{(2)}{m}-1},$$

$$A_{31}=r_1^{\overset{(1)}{m}}\,J_{P1}\left(r_1\frac{\omega}{\beta_{01}}\right),\qquad A_{32}=r_1^{\overset{(1)}{m}}J_{-P1}\left(r_1\frac{\omega}{\beta_{01}}\right),$$

$$A_{33}=-r_1^{\overset{(2)}{m}}J_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right),\qquad A_{31}=-r_1^{\overset{(2)}{m}}J_{P2}\left(r_1\frac{\omega}{\beta_{02}}\right),$$

$$A_{41}=0\qquad A_{42}=0$$

$$A_{43}=R_0 J^1{}_{P2}\left(R_0\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{P2}\left(R_0\frac{\omega}{\beta_{02}}\right),$$

$$A_{44}=R_0 J^1{}_{-P2}\left(R_0\frac{\omega}{\beta_{02}}\right)+\left(\overset{(2)}{m}-1\right)J_{-P2}\left(R_0\frac{\omega}{\beta_{02}}\right),$$

7. Three layered model (with solid core)

In the previous section we have obtained the frequency equation for three layered earth model, with inner sphere as liquid one. In this section frequency equation for three layered model with the inner sphere solid one, has been obtained as follows :

Boundary conditions are :

$$[T_{r\phi}]_{r=r_0} = 0 \tag{45a}$$

$$\left.\begin{matrix}[T_{r\phi}]_1 = [T_{r\phi}]_2 \\ \omega_1 = \omega_2\end{matrix}\right] \text{at } r = r_1 \tag{45b}$$

$$[T_{r\phi}]_2 = [T_{r\phi}]_3 \text{ and } \omega_2 = \omega_3 \text{ at } r = R_0 \tag{45c}$$

Eqns. (43), (44) and (45) give Eqns. (41), (42) and (43) and the equations

$$\mu_{02}\, p_2\, (R_0)^{\overset{(2)}{m}-1}\Big\{A_2\Big[R_0 J^1{}_{P_2}\Big(R_0\frac{\omega}{\beta_{02}}\Big)+\Big(\overset{(2)}{m}-1\Big)J_{P_2}\Big(R_0\frac{\omega}{\beta_{02}}\Big)\Big]+B_2\Big[R_0 J^1{}_{-P_2}\Big(R_0\frac{\omega}{\beta_{02}}\Big)+\Big(\overset{(2)}{m}-1\Big)$$

$$J_{-P_2}\Big(R_0\frac{\omega}{\beta_{02}}\Big)\Big]\Big\} = \mu_{03}\, p_3\,(R_0)\, R_0^{\overset{(3)}{m}-1}\Big[R_0 J^1{}_{P_3}\Big(R_0\frac{\omega}{\beta_{03}}\Big)+\Big(\overset{(3)}{m}-1\Big)J_{P_3}\Big(R_0\frac{\omega}{\beta_{03}}\Big)\Big] \tag{46}$$

$$R_0^{\overset{(2)}{m}}\Big[A_2 J_{P_2}\Big(R_0\frac{\omega}{\beta_{02}}\Big)+B_2 J_{-P_2}\Big(R_0\frac{\omega}{\beta_{02}}\Big)\Big] = A_3\, J_{P_3}\Big(R_0\frac{\omega}{\beta_{03}}\Big)R_0^{\overset{(3)}{m}} \tag{47}$$

Eliminating the unknown constants A_1, B_1, A_2, B_2 and A from the Eqns. (41), (42), (43) and (45) and (47) we get

$$|B| = 0$$

where the elements of the matrix B are given by

$$B_{11} = r_0 J^1{}_{P_1}\Big(r_0\frac{\omega}{\beta_{01}}\Big)+\Big(m^{(1)}-1\Big)J_{P_1}\Big(r_0\frac{\omega}{\beta_{01}}\Big),$$

$$B_{12} = r_0 J^1{}_{-P_1}\Big(r_0\frac{\omega}{\beta_{01}}\Big)+\Big(\overset{(1)}{m}-1\Big)J_{-P_1}\Big(r_0\frac{\omega}{\beta_{01}}\Big),$$

$$B_{13} = 0\,, \qquad B_{14} = 0\,, \qquad B_{15} = 0\,,$$

$$B_{21} = \mu_{01}\, p_1(r_1)\Big[r_1 J^1{}_{P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)+\Big(\overset{(1)}{m}-1\Big)J_{P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)\Big]r_1^{\overset{(1)}{m}-1},$$

$$B_{22} = \mu_{01}\, p_1(r_1)\Big[r_1 J^1{}_{-P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)+\Big(m^{(1)}-1\Big)J_{-P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big),$$

$$B_{13} = 0, \qquad B_{14} = 0, \qquad B_{15} = 0,$$

$$B_{21} = \mu_{01}\, p_1(r_1)\Big[r_1 J^1{}_{P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)+\Big(\overset{(1)}{m}-1\Big)J_{P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)\Big]r_1^{\overset{(1)}{m}-1},$$

$$B_{22} = \mu_{01}\, p_1(r_1)\Big[r_1 J^1{}_{-P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)+\Big(m^{(1)}-1\Big)J_{-P_1}\Big(r_1\frac{\omega}{\beta_{01}}\Big)\Big]r_1^{\overset{(1)}{m}-1},$$

$$B_{23} = -\,\mu_{02}\, p_2(r_2)\Big[r_1 J^1{}_{P_2}\Big(r_1\frac{\omega}{\beta_{02}}\Big)+\Big(m^{(2)}-1\Big)J_{-P_2}\Big(r_1\frac{\omega}{\beta_{02}}\Big)\Big]r_1^{\overset{(2)}{m}-1},$$

$$B_{24} = -\,\mu_{02}\, p_2(r_2)\Big[r_1 J^1{}_{-P_2}\Big(r_1\frac{\omega}{\beta_{02}}\Big)+\Big(m^{(2)}-1\Big)J_{-P_2}\Big(r_1\frac{\omega}{\beta_{02}}\Big)\Big]r_1^{\overset{(2)}{m}-1},\ B_{25}=0$$

$$B_{31} = r_1^{m^{(1)}} J_{P1}\left(r_1 \frac{\omega}{\beta_{01}} \right) ,$$

$$B_{32} = r_1^{m^{(1)}} J_{-P1}\left(r_1 \frac{\omega}{\beta_{01}} \right) ,$$

$$B_{33} = -r_1^{m^{(2)}} J_{P2}\left(r_1 \frac{\omega}{\beta_{02}} \right) ,$$

$$B_{34} = -r_1^{m^{(2)}} J_{-P2}\left(r_1 \frac{\omega}{\beta_{02}} \right) ,$$

$$B_{35} = 0 , \qquad B_{41} = 0 , \qquad B_{42} = 0 ,$$

$$B_{43} = \mu_{02}\, p_2 (R_0) \left[R_0 \; J^1{}_{P2} \left(R_0 \frac{\omega}{\beta_{02}} \right) + \right.$$

$$\left. + \left(m^{(2)} - 1 \right) J_{P2} \left(R_0 \frac{\omega}{\beta_{02}} \right) \right] R_0^{m^{(2)}-1},$$

$$B_{44} = \mu_{02}\, p_2 (R_0) \left[R_0 J^1{}_{-P2} \left(R_0 \frac{\omega}{\beta_{02}} \right) + \right.$$

$$\left. + \left(m^{(2)} - 1 \right) J_{-P2} \left(R_0 \frac{\omega}{\beta_{02}} \right) \right] R_0^{m^{(2)}-1},$$

$$B_{45} = - \mu_{03}\, p_3 (R_0) \left[R_0 J^1{}_{P3} \left(R_0 \frac{\omega}{\beta_{03}} \right) + \right.$$

$$+ \left(m^{(3)} - 1 \right) J_{P3} \left(R_0 \frac{\omega}{\beta_{03}} \right) R_0^{m^{(3)}-1},$$

$$B_{51} = 0 , \qquad B_{52} = 0 ,$$

$$B_{53} = R_0^{m^{(2)}} J_{P2}\left(R_0 \frac{\omega}{\beta_{02}} \right) ,$$

$$B_{54} = R_0^{m^{(2)}} J_{-P2}\left(R_0 \frac{\omega}{\beta_{02}} \right) ,$$

$$B_{55} = - R_0^{m^{(3)}} J_{P3}\left(R_0 \frac{\omega}{\beta_{03}} \right)$$

8. Conclusions

We have assumed that the lateral variation at all depths is same whereas the radial variation is different. Shear wave velocity is constant in each layer but it varies from layer to layer. Wave equation can also be separated if the ratio is proportional to r^2. But this assumption gives us the velocity in the earth model to be decreasing with depth which is physically unrealistic. This assumption has been by Singh *et al.* (1976b). Frequency equation has been obtained by extending Thomson-Haskell method to radially and laterally heterogeneous spherical model. Special cases of two layered and three layered models have been discussed.

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