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Relationship between horizontal spiralling wind field and pressure field

AIR CMDE. S. LAKSHMINARAYANAN

Indian Air Force HQ, New Delhi

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सार — पवन क्षेत्र से संबद्ध सर्पिलन पवन क्षेत्र, जैसा कि चक्रवात में होता है, को क्षेत्रीय समाकलन और तुल्य कोणों की ओर समाकलन से ज्ञात किया गया है। इसके बाद, ऊर्ध्वाधर गति के पद को ज्ञात किया गया है। यह दर्शाया गया है कि चक्रवात के केन्द्र में ऊर्ध्वाधर अधोमुखी गति उस समय होती है जब चक्रवात अपेक्षाकृत अधिक तीव्र होता है जिसके लक्षण यह हैं कि तीव्र पवन अधिकतम वृत्ताकार पवन की परिधि पर होती है और/अथवा अधिकतम वृत्ताकार पवन की परिधि का संकुचन होता है।

ABSTRACT. Pressure field associated with a spiralling wind field as in a cyclone is obtained by areal integration and also by integration along isogons. Further, expression for vertical motion is obtained. Vertical downward motion is shown to occur in the eye of the cyclone when the cyclone is relatively intense characterised by strong winds on the ring of maximum wind and/or contraction of ring of maximum wind.

1. Introduction

1.1. An important problem in meteorology can be stated as follows :

Given a wind field, obtain corresponding pressure field from the equation of motion for horizontal wind vector **V** as in (1).

$$\rho\left\{\frac{\partial \mathbf{V}}{\partial t} + w\,\frac{\partial \mathbf{V}}{\partial z} + (\mathbf{V}.\,\nabla)\,\mathbf{V} + (\mathbf{k}\times\mathbf{V})\,f + \right.$$

$$\left. + K\,\mathbf{V} + 2\,\Omega\cos\Phi\,w\,\mathbf{i}\right\} = -\nabla p \qquad (1)$$

Here K is the coefficient of friction. In this paper we take frictional term as $K\mathbf{V}$. But for frictional term, the equation of motion (1) is the same as 8.61 p. 253, *Dynamic Meteorology and Weather Forecasting* by Godske *et al.* (1957). Geostrophic and gradient winds extensively used in meteorology are none other than solutions of (1) under various constraints. Isobars are same as streamlines in both the cases. Since cross isobaric winds are conspicuous in cyclones with closed isobars and further vertical motion associated with vertical shear are also important in cyclones, solution of (1) merits a *de novo* attack. The objective of this paper are (*i*) to obtain relation between cyclone wind field and pressure field and (*ii*) vertical motion in a cyclone.

1.2. *Nomenclature*

$\mathbf{V} = u\,\mathbf{i} + v\,\mathbf{j} = V(\cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j})$, Horizontal wind vector

- ψ : direction reckoned positive counterclockwise.
- s : distance along streamline
- n : distance along normal to streamline
- n_1 : distance along isogon : positive along $(\nabla\psi\times\mathbf{k})$ vector line
- r : radial distance in polar coordinate
- θ : angle in polar coordinate
- $\mathbf{n}$: unit outward drawn normal vector

$$\nabla.\,\mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = V\frac{\partial \psi}{\partial n} + \frac{\partial V}{\partial s} \quad \text{divergence}$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = V\,\frac{\partial \psi}{\partial s} - \frac{\partial V}{\partial n} \quad \text{vorticity}$$

$$L_1 = 4\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right) = 4|\nabla\psi|\,\frac{\partial}{\partial n_1}\,\frac{V^2}{2}$$

Kinematical determinant of **V**

$$L_2 = L_1 - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2$$

$\partial\psi/\partial s$ if positive, streamline is counterclockwise curved and $\partial\psi/\partial s$ if negative, clockwise curved.

$\partial\psi/\partial n$ if positive, streamline are difluenting and if negative, they are confluenting.

θ_{is} angle between isogon and streamline

$\hat{\theta}_{pis}$ angle between isobar and isogon.

2. Characteristics of spiralling wind

2.1. Streamlines in a spiralling wind field terminate at or originate from a point called the spiral centre and the curvature is counterclockwise/clockwise. At the spiral centre, the speed is zero. Confining our attention to cyclone circulation following characteristics are noted :

(*i*) Streamlines spiral inward and terminate at cyclone centre where speed is zero.

(*ii*) Curvature of streamlines is counterclockwise/clockwise in northern/southern hemisphere. Hence $f(\partial\psi/\partial s)$ is positive, where f is coriolis parameter and $\partial\psi/\partial s$ is curvature.

(*iii*) Since streamlines spiral inward, they are confluenting, *i.e.*, $\partial\psi/\partial n$ is negative.

(*iv*) The spiral centre is enclosed by a ring of maximum wind (RMW).

(*v*) Isogons originate from the cyclone centre.

Going round the cyclone centre counterclockwise, the direction ψ continuously increases and changes by $+2\pi$ on making one complete circuit. The direction ψ is the angle wind vector makes with x-axis and is reckoned positive counterclockwise. Meteorological convention is to reckon direction positive clockwise. Hence it must be noted that direction will continuously decrease and change by -2π on making a counterclockwise circuit round the cyclone centre if meteorologically reckoned direction is used.

2.2. Vector lines $(\nabla\psi \times \mathbf{k})$ are same as isogons. In this paper, distance n_1 along isogon is reckoned positive in the direction of $(\nabla\psi\times\mathbf{k})$. The isogon characteristic (v) can be mathematically expressed as "Isogons, *i.e.*, $(\nabla\psi \times \mathbf{k})$ lines radiate out of cyclone centre and $\oint (\mathbf{n} \times \nabla\psi)\, dl = +2\pi\,\mathbf{k}$ where contour encloses cyclone centre". In this connection, please see Theorem III of Lakshminarayanan (1978).

2.3. The spiralling wind field $\mathbf{V}$ of a cyclone can be modelled as $\mathbf{V} = A\nabla\phi + B\nabla\phi \times \mathbf{k}$ as in Lakshminarayanan (1975) paper. Here $\phi = \exp[-(x^2+y^2)/2\sigma^2]$, σ is the radius of ring of maximum wind. Taking A and B as positive, we get spiralling-in counterclockwise curved streamline pattern as in northern hemispheric cyclone. $A\nabla\phi = \mathbf{V}_i$ gives inflow feature and $B\nabla\phi \times \mathbf{k} = \mathbf{V}_R$ gives rotational feature.

$$V_i = \left(\frac{A}{\sigma}\right)\left(\frac{r}{\sigma}\right)\phi \; ; V_R = \left(\frac{B}{\sigma}\right)\left(\frac{r}{\sigma}\right)\phi \; ;$$

$$V = \frac{(A^2+B^2)^{\frac{1}{2}}}{\sigma}\left(\frac{r}{\sigma}\right)\phi$$

$$(V_i)_{max} = \left(\frac{A}{\sigma}\right)e^{-1/2}; \; (V_R)_{max} = \frac{B}{\sigma}\, e^{-1/2} \; ;$$

$$(V)_{max} = \frac{(A^2+B\;)^{\frac{1}{2}}}{\sigma}\, e^{-1/2}$$

$$\nabla.\mathbf{V} = -A\left(2-\frac{r^2}{\sigma^2}\right)\phi \; ; \zeta = B\left(2-\frac{r^2}{\sigma^2}\right)\phi$$

We will use the model in calculating vertical motion and pressure field at a later stage.

3. Isogon coordinates

3.1. Cartesian (x, y), polar (r, θ) and streamline coordinates (s, n) are commonly used in meteorology. A better analytical insight into cyclone features is provided if we use isogons, *i.e.*, $(\nabla\psi \times \mathbf{k})$ vector lines and normal to isogon, *i.e.*, $(\nabla\psi \times \mathbf{k})$ lines as a coordinate system.

3.2. $\mathbf{V} = \mathbf{V}_{IS} + \mathbf{V}_{NIS}$ where $\mathbf{V}_{IS}$ is component along isogon and $\mathbf{V}_{NIS}$ component normal to isogon. Suffix 'IS' and 'NIS' refer to isogon and normal to isogon. Since $V_{IS} = V(\partial\psi/\partial n)\,|\nabla\psi|^{-1}$ and $V_{NIS} = V(\partial\psi/\partial s)\,|\nabla\psi|^{-1}$, we note immediately in the case of cyclone that $\mathbf{V}_{IS}$ gives inflow features and $\mathbf{V}_{NIS}$ rotational features.

3.3. To avoid *in situ* derivation and facilitate ease of reference, we list the components in Table 1.

Advection term $(\mathbf{V}.\nabla)\,\mathbf{V}$ can also be written as :

$$\begin{aligned}(\mathbf{V}.\nabla)\,\mathbf{V} &= \frac{\partial V}{\partial s}\,\mathbf{V} + V\frac{\partial\psi}{\partial s}\,(\mathbf{k}\times\mathbf{V}) \\ &= (\nabla.\mathbf{V})\,\mathbf{V} - V^2(\nabla\psi\times\mathbf{k}) \\ &= \nabla\frac{V^2}{2} + (\mathbf{k}\times\mathbf{V})\,\zeta\end{aligned}$$

If $\mathbf{V}$ is irrotational, *i.e*, $\nabla.\mathbf{V} = 0$ at all points, then $(\mathbf{V}.\nabla)\,\mathbf{V} = \nabla V^2/2$

If $\mathbf{V}$ is solenoidal, *i.e.*, $\zeta = 0$ at all points, then $(\mathbf{V}.\nabla)\,\mathbf{V} = -V^2(\nabla\psi \times \mathbf{k})$

We will be using the advection term $(\mathbf{V}.\nabla)\,\mathbf{V}$ expressed in terms of $\nabla.\mathbf{V}$ and ζ at a later context. The term $2\,\Omega\,\cos\Phi\, w\,\mathbf{i}$ is small and is customarily neglected in atmospheric cyclone problem.

3.4. Multiplying $\nabla.\mathbf{V} = V(\partial\psi/\partial n) + \partial V/\partial s$ by $V(\partial\psi/\partial s)$ and vorticity $\zeta = V(\partial\psi/\partial s) - \partial V/\partial n$ by $V(\partial\psi/\partial n)$ and subtracting, we get :

$$(\nabla.\mathbf{V})\left(V\frac{\partial\psi}{\partial s}\right) - (\zeta)\left(V\frac{\partial\psi}{\partial n}\right) = \left(\nabla\frac{V^2}{2}\right).(\nabla\psi)$$

TABLE 1

Nomenclature	Term	s	n	Along isogon	Normal to isogon
(1) Local change	$\frac{\partial \mathbf{V}}{\partial t}$	$\frac{\partial}{\partial t}\log V$	$\frac{\partial \psi}{\partial t} V$	$\frac{\partial}{\partial t}\log V\, V_{\mathrm{IS}} - \frac{\partial \psi}{\partial t} V_{\mathrm{NIS}}$	$\frac{\partial}{\partial t}\log V\, V_{\mathrm{NIS}} + \frac{\partial \psi}{\partial t} V_{\mathrm{IS}}$
(2) Vertical motion with vertical shear	$w\frac{\partial \mathbf{V}}{\partial z}$	$w\frac{\partial}{\partial z}\log V$	$w\frac{\partial \psi}{\partial z} V$	$\left(\frac{\partial}{\partial z}\log V\, V_{\mathrm{IS}} - \frac{\partial \psi}{\partial z} V_{\mathrm{NIS}}\right) w$	$\left(\frac{\partial}{\partial z}\log V\, V_{\mathrm{NIS}} + \frac{\partial \psi}{\partial z} V_{\mathrm{IS}}\right) w$
(3) Advection term	$(\mathbf{V}.\nabla)\mathbf{V}$	$\frac{\partial V}{\partial s} V$	$V^2\frac{\partial \psi}{\partial s}$	$\frac{\partial V}{\partial s} V_{\mathrm{IS}} - V\frac{\partial \psi}{\partial s} V_{\mathrm{NIS}}$	$\frac{\partial V}{\partial s} V_{\mathrm{NIS}} + V\frac{\partial \psi}{\partial s} V_{\mathrm{IS}}$
(4) Coriolis term	$(\mathbf{k}\times\mathbf{V})f$	0	fV	$-fV_{\mathrm{NIS}}$	$+fV_{\mathrm{IS}}$
(5) Frictional term	$K\mathbf{V}$	$K V$	0	$K V_{\mathrm{IS}}$	$K V_{\mathrm{NIS}}$
(6) Vertical motion with $2\Omega\cos\Phi$	$2\Omega\cos\Phi\, w\,\mathbf{i}$	$2\Omega\cos\Phi\, w\cos\psi$	$2\Omega\cos\Phi\, w\sin\psi$	$2\Omega\cos\Phi\, w\frac{\partial \psi}{\partial y}\,\lvert\nabla\psi\rvert^{-1}$	$-2\Omega\cos\Phi\, w\frac{\partial \psi}{\partial x}\,\lvert\nabla\psi\rvert^{-1}$

We can use this identity to express divergence in terms of vorticity and *vice versa*. Further in well developed cyclones, isogons are radial lines and isopleths of speed are circles. Hence, $(\nabla V^2/2).(\nabla\psi) = 0$. In such a case :

$$\frac{\nabla.\mathbf{V}}{\zeta} = \frac{V\frac{\partial\psi}{\partial n}}{V\frac{\partial\psi}{\partial s}} = \tan\theta_{\mathrm{is}}$$

where θ_{is} is the angle between isogon and streamline. If spiralling pattern is preserved, *i.e.*, θ_{is} does not change with time, then intensification of cyclone characterised by increase of speed and/or contraction of RMW, can be expected to increase both convergence and vorticity. In short, it is realistic to assume increase of convergence and vorticity in the intensification stage of cyclone.

THEOREM VII

"Given that (*i*) isogon and isopleths of speed are orthogonal and (*ii*) the angle between isogon and streamline is unchanged, then increase in magnitude of divergence requires increase in magnitude of vorticity and likewise decrease".

Proof

$$\left(\nabla.\mathbf{V}\right)\left(V\frac{\partial\psi}{\partial s}\right)-\left(\zeta\right)\left(V\frac{\partial\psi}{\partial n}\right) \equiv \left(\nabla\frac{V^2}{2}\right)\cdot\left(\nabla\psi\right)\equiv 0$$

$$\Rightarrow \frac{\left(\nabla.\mathbf{V}\right)}{\left(\zeta\right)} = \frac{\left(V\frac{\partial\psi}{\partial n}\right)}{\left(V\frac{\partial\psi}{\partial s}\right)} = \tan\left(\theta_{is}\right) = \text{constant}$$

Hence the theorem.

Comment

In the central region of cyclone, isogons and isopleths of speed are very nearly orthogonal and this property appears to exist throughout the life span of cyclone. The theorem explains the reason why convergence and vorticity together have higher values as the cyclone intensifies when speed on RMW increases and/or the radius of RMW contracts. We will be using this while discussing vertical motion in intense stage of cyclone.

3.5. $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, $\frac{\partial^2 p}{\partial x^2}$, $\frac{\partial^2 p}{\partial y^2}$ and $\frac{\partial^2 p}{\partial x\partial y}$ are necessary to determine the occurrence of maximum, minimum, minimaximum of pressure at a point. Instead, we propose a simple rule as in Table 2 for determination of the same with first derivate. Let n_1 be distance measured in lines radiating from a point. Evaluate $\partial p/\partial n_1$ at all points in the immediate neighbourhood surrounding the point. The terms on L.H.S. of equation of motion (1) are individually investigated to determine what will be the pressure field each term will favour at the cyclone centre, *viz.*, a maximum, minimum, an isobar or no pressure field. We will be using this rule in that context.

4. Pressure field

4.1. A spiralling wind field can be thought of as an addition of an irrotational field giving inflow/outflow feature and a solenoidal field giving rotational feature. To gain analytical insight, we investigate the pressure field associated with irrotational, solenoidal wind field and spiralling wind field satisfying $\rho\,(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p$.

4.2.1. *Irrotational field*

THEOREM 1

Given an irrotational horizontal wind field $\mathbf{V}$ satisfying $\rho\,(\mathbf{V}.\nabla)\,\mathbf{V} = -\nabla p$, then isopleths

TABLE 2

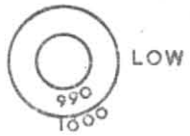
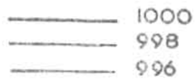
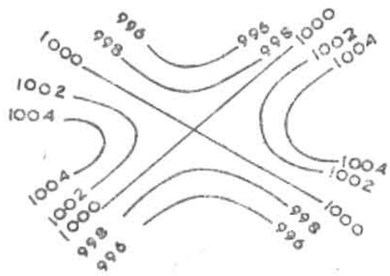
$\partial p/\partial n_1$ characteristic	Pressure field characteristic	Schematic illustration
(*i*) $\partial p/\partial n_1$ is positive at all points in the neighbourhood	Minimum pressure	
(*ii*) $\partial p/\partial n_1$ is negative at all points in the neighbourhood	Maximum pressure	
(*iii*) Making a circuit around the point, if we find $\partial p/\partial n_1$ is zero, positive, zero, and negative	An isobar passes through the point	
(*iv*) Making a circuit around the point, if we find, $\partial p/\partial n_1$ is zero, positive, zero, negative, zero, positive, zero and negative	A minimaximum of pressure	
(*v*) $\partial p/\partial n_1$ is zero at all points	Pressure is constant	

Fig. 1

of speed is the same as isobars and isopleth of density. Further pressure maximum will occur at speed minimum and pressure minimum at speed maximum.

Proof

Since vorticity ζ vanishes in an irrotational field, $(\mathbf{V}.\nabla)\,\mathbf{V} = \nabla V^2/2$. Therefore $\rho\,\nabla V^2/2 = -\nabla p$. Hence isopleth of speed is the same as isobar as well as isopleth of density and pressure maximum occurs at speed minimum and pressure minimum at speed maximum.

Comment

Pressure maximum is necessary at the point of origination of streamlines of an outflow as well as at the point of termination of an inflow as illustrated in Fig. 1 since in both the cases the speed is a minimum at the point of origination as well as termination of streamlines.

4.2.2. *Solenoidal field*

THEOREM 2

Given a solenoidal horizontal wind vector $\mathbf{V}$ satisfying $\rho\,(\mathbf{V}.\,\nabla)\,\mathbf{V} = -\nabla p$, isobars are orthogonal to isogons. Further a minimum pressure must occur at the point enclosed by streamlines.

Proof

Since divergence $(\nabla.\mathbf{V})$ vanishes in a solenoidal field, $(\mathbf{V}.\,\nabla)\,\mathbf{V} = -V^2\,(\nabla\psi \times \mathbf{k})$. Therefore, $\rho\,V^2\,(\nabla\psi \times \mathbf{k}) = +\nabla p$. Hence isobars are orthogonal to isogons. $(\nabla\psi \times \mathbf{k})$ vector lines which are isogons radiate out of the point enclosed by streamlines. $\partial p/\partial n_1 = \rho\,V^2\,|\nabla\psi|$. Since $\partial p/\partial n_1$ is positive at all points surrounding the point enclosed by streamlines, the pressure at that point is a minimum.

Comment

Pressure minimum occurs whether the streamlines are counterclockwise or clockwise since in both the cases $(\nabla\psi \times \mathbf{k})$ lines radiate outward only.

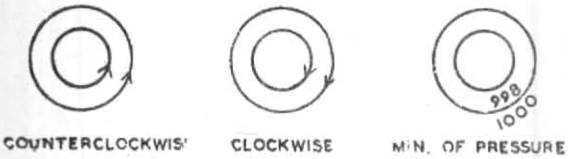


Fig. 2

4.2.3. *Spiral field*

Cyclone is characterised by both inflow and rotational features. Inflow favours a maximum and rotation a minimum. Since rotation in predominent compared to inflow in atmospheric cyclones, a minimum occurs. In the irrotational cases as well as rotational cases, $(\mathbf{V}.\nabla)\mathbf{V}$ lines are compatible with ∇p vector lines. Now we study vector lines of $(\mathbf{V}.\nabla)\mathbf{V}$ for a spiralling $\mathbf{V}$ field. As a preliminary we invoke Therorem IV (Lakshminarayanan 1978) defining spiralling property. Kinematical determinant of $\mathbf{V} = u\,\mathbf{i} + v\,\mathbf{j}$ is defined as $L_1 = 4\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$ and $L_2 = L_1 - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2$. Streamlines/vector lines of $\mathbf{V}$ in the immediate neighbourhood of a point where $\mathbf{V} = 0$, L_1 and L_2 are positive will be spiral shaped or closed lines.

The kinematical determinant of :

(1) $\left(\mathbf{V}.\nabla\right)\mathbf{V} = \frac{L_1^2}{4}$ at $\mathbf{V} = 0$

(2) $\left(\mathbf{V}.\nabla\right)\mathbf{V} - \left[\nabla\cdot\left\{(\mathbf{V}\cdot\nabla)\mathbf{V}\right\}\right]^2$

$$= L_2\left(\nabla.\mathbf{V}\right)^2 \text{ at } \mathbf{V} = 0$$

(3) $\nabla p = 4\left\{\frac{\partial^2 p}{\partial x^2}\frac{\partial^2 p}{\partial y^2} - \left(\frac{\partial^2 p}{\partial x \partial y}\right)^2\right\}$

$$= 4 \text{ Hessian of } p$$

(4) $\nabla p - \left\{\nabla\cdot\left(\nabla p\right)\right\}^2$

$$= -\left(\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2}\right)^2 - 4\left(\frac{\partial^2 p}{\partial x \partial y}\right)^2 = -\text{ve}$$

THEOREM 3

Given a spiralling wind field $\mathbf{V}$ such that $\mathbf{V}=0$ and L_1 and L_2 are positive at spiral centre, then $(\mathbf{V}.\nabla)\mathbf{V}$ will be spiral shaped. Given a pressure field such that Hessian of p exists at points where $\nabla p = 0$, then ∇p vector lines will not be closed and will not be spiralling.

Proof

Using kinematical determinant of $(\mathbf{V}.\nabla)\mathbf{V}$ and ∇p and kinematical determinant *minus* (divergence)^2 as in Theorem IV(c), this theorem is proved.

Comment

(*i*) In the case of a cyclone, $\rho\,(\mathbf{V}.\nabla)\mathbf{V} \neq -\nabla p$ since $\rho(\mathbf{V}\cdot\nabla)\,\mathbf{V}$ vector lines are spiral shaped and ∇p lines cannot be spiral shaped. Further $\rho\,(\mathbf{k}\times\mathbf{V})f$ and $K\mathbf{V}$ are also spiral shaped. Hence,

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + w\frac{\partial \mathbf{V}}{\partial z}\right)$$

must exist and ensure vector line of left hand side of Eqn. (1) to be compatible with vector lines of $-\nabla p$.

4.2.4. *Pressure field*

∇p is an irrotational vector. Its vector lines cannot be spiral shaped or closed. That ∇p vector lines cannot be closed lines can also be established by using Stoke's Theorem as well.

THEOREM 4

Closed ∇p vector lines without change of direction is impossible.

Proof

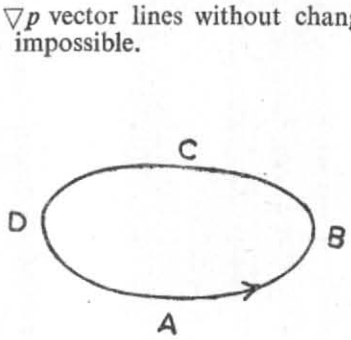


Fig. 3

If possible, let ABCDA as in Fig. 3 be a closed ∇p vector line without change in direction. Using Stokes Theorem,

$$\iint\left(\nabla\times\nabla p\right)d\mathbf{S} = \oint\left(\nabla p\right)\cdot\left(d\mathbf{l}\right) = \oint|\nabla p|\,|d\mathbf{l}|$$

where, area of integration is enclosed by ABCDA, $d\mathbf{S}$ is an element of area directed upwards and $d\mathbf{l}$ is an element of length parallel to ∇p. We note that $\nabla\times\nabla p \equiv 0$ and $(\nabla p).(d\mathbf{l}) = |\nabla p|\,|d\mathbf{l}|$ = a positive quantity. Since,

$\iint\left(\nabla\times\nabla p\right)\cdot d\mathbf{S} = 0$ and $\oint|\nabla p|\,|d\mathbf{l}|$ is a positive quantity, closed ∇p vector line is impossible.

Comment

$\oint(\nabla p).(d\mathbf{l}) = 0$ where the contour is any closed curve. We will make use of it to evaluate vertical motion in cyclone.

4.3. *Closed isobars*

Isobars enclose a pressure minimum in the case of cyclones. In such cases, we can obtain pressure by areal integration.

THEOREM 5

Given a spiral circulation with a spiral centre enclosed by isobars, then

$$\iint \left(\frac{\partial p}{\partial s}\frac{\partial \psi}{\partial n} - \frac{\partial p}{\partial n}\frac{\partial \psi}{\partial s}\right) dS = 2\pi (p_2 - p_1)$$

where, (*i*) area of integration is bounded by closed isobars p_1 and p_2

(*ii*) p_1 is inside p_2

(*iii*) spiral centre is inside p_1

Proof

$$\nabla p \times \nabla \psi = \nabla \times (p \nabla \psi)$$

$$\iint (\nabla p \times \nabla \psi)\, d\mathbf{S} = \oint p\, (\mathbf{n} \times \nabla \psi)\, dl$$

$$= p \oint (\mathbf{n} \times \nabla \psi)\, dl$$

where area of integration is enclosed by isobars. Invoking Lakshminarayanan's Theorem III as given 2.2, we get $\oint (\mathbf{n} \times \nabla \psi)\, dl = +2\pi \mathbf{k}$. Hence,

$$\iint (\nabla p \times \nabla \psi)\, \mathbf{k}\, d\mathbf{S} = \iint \left(\frac{\partial p}{\partial s}\frac{\partial \psi}{\partial n} - \frac{\partial p}{\partial n}\frac{\partial \psi}{\partial s}\right) d\mathbf{S} = 2\pi (p_2 - p_1)$$

Comment (*i*)

If there is no spiral centre and more specifically there is no point where $\mathbf{V} = 0$ and isobars are closed, then

$$\iint \left(\frac{\partial p}{\partial s}\frac{\partial \psi}{\partial n} - \frac{\partial p}{\partial n}\frac{\partial \psi}{\partial s}\right) d\mathbf{S} = 0$$

Comment (*ii*)

From equation of motion (1), we get

$$\frac{\partial p}{\partial s} = -\rho V \left(K + \frac{d}{dt} \log V\right)$$

and $\quad \dfrac{\partial p}{\partial n} = -\rho V \left(f + \dfrac{d\psi}{dt}\right)$. Hence,

$$\iint \rho \left\{ \left(f + \frac{d\psi}{dt}\right) V \frac{\partial \psi}{\partial s} - \left(\quad + \frac{d \log V}{dt}\right) \times V \frac{\partial \psi}{\partial n} \right\} d\mathbf{S} = 2\pi (p_2 - p_1) \qquad (2)$$

Comment (*iii*)

$\dfrac{\partial p}{\partial s}$ is rate of change of pressure along streamline.

$\dfrac{\partial p}{\partial s}$ negative $\Rightarrow \left(K + \dfrac{d \log V}{dt}\right)$ positive

$\Rightarrow$ (Cross isobaric flow into a relatively lower pressure)

$\dfrac{\partial p}{\partial s}$ positive $\Rightarrow \left(K + \dfrac{d}{dt} \log V\right)$ negative

$\Rightarrow$ (Cross isobaric flow into a relatively higher pressure)

$\dfrac{\partial p}{\partial s} = 0 \Rightarrow \left(K + \dfrac{d \log V}{dt}\right) = 0$

$\Rightarrow$ (Isobar and streamline are the same)

$\dfrac{\partial p}{\partial n}$ is rate of change normal to streamline

$\dfrac{\partial p}{\partial n}$ negative $\Rightarrow \left(f + \dfrac{d\psi}{dt}\right)$ positive

$\Rightarrow$ (Relatively lower pressure to left of streamline)

$\dfrac{\partial p}{\partial n}$ positive $\Rightarrow \left(f + \dfrac{d\psi}{dt}\right)$ negative

$\Rightarrow$ (Relatively higher pressure to left of streamline)

$\dfrac{\partial p}{\partial n} = 0 \Rightarrow \left(f + \dfrac{d\psi}{dt}\right) = 0$

$\Rightarrow$ (Streamline and isobar are orthogonal)

We immediately note that if we constrain that $(f + d\psi/dt)$ is positive in northern hemisphere and negative in southern hemisphere and $K + d(\log V)/dt$ is positive in both hemispheres, we get Buys Ballot's Law.

Comment (*iv*)

The only constraints are the existence of spiral centre and isobars enclosing it. Hence this relation has applicability especially for unsteady and tilted system with growth/decay, vertical motion and vertical shear, *i.e.*, where $(\partial \mathbf{V}/\partial t) + w(\partial \mathbf{V}/\partial z)$ term exists.

4.4. *Deepening of pressure system*

Let p_0 be the outermost isobar and p_c be the pressure at cyclone centre. Noting $\nabla p \times \nabla \psi = |\nabla p|\,|\nabla \psi| \sin \theta_{pis}$ we get from Theorem 5 :

$$2\pi (p_0 - p_c) = \iint \rho \left\{ \left(f + \frac{d\psi}{dt}\right)^2 + \left(\quad + \frac{d \log V}{dt}\right)^2 \right\}^{\frac{1}{2}} V \sin \theta_{pis} \, |\nabla \psi| \; d\mathbf{S} \qquad (3)$$

where θ_{pis} is the angle between isobar and isogon. Now we assume $|\nabla \psi| = 1/r$ and outermost isobar p_0 remains the same as cyclone intensifies. Deepening pressure system is indicated by relatively very low values of p_c. Deepening, therefore, will be associated with increase of value of terms inside the integral. Hence, (*i*) increase of value of speed: the mean speed increasing with time will indicate deepening.

(*ii*) Increase of $\sin\theta_{pis}$, *i.e.*, isobars and isogon tending to become orthogonal.

(*iii*) Increase of

$$\left\{\left(f+\frac{d\psi}{dt}\right)^2+\left(K+\frac{d\log V}{dt}\right)^2\right\}^{\frac{1}{2}}$$

The role of $\left\{\left(f+\frac{d\psi}{dt}\right)^2+\left(K+\frac{d\log V}{dt}\right)^2\right\}^{\frac{1}{2}}$ requires specific case studies. However, the increase of speed and isobars tending to become orthogonal to isogon in the context of deep pressure systems associated with cyclone are observationally seen. A schematic diagram illustrates the near orthogonal property between isogons and isobars.

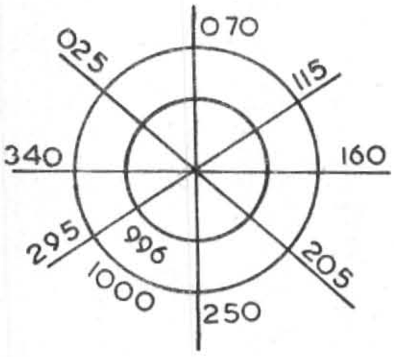


Fig. 4

4.5. *Integration along isogon*

$\partial p/\partial n_1$ is rate of variation of pressure along isogon. Hence integrating isogon from cyclone centre where pressure is p_c, we get

$p-p_c=\int_c \frac{\partial p}{\partial n_1}dn_1$. From Table 1, we get:

$$\left.\begin{aligned}\frac{\partial p}{\partial n_1} = -\rho\frac{\partial \log V}{\partial t}\, V\frac{\partial\psi}{\partial n}\,|\nabla\psi|^{-1}\\ +\rho\frac{\partial\psi}{\partial t}\, V\frac{\partial\psi}{\partial s}\,|\nabla\psi|^{-1}\end{aligned}\right\}$$

(Local change term)

$$\left.\begin{aligned}-\rho w\frac{\partial \log V}{\partial z}\, V\frac{\partial\psi}{\partial n}\,|\nabla\psi|^{-1}\\ +\rho w\frac{\partial\psi}{\partial z}\, V\frac{\partial\psi}{\partial s}\,|\nabla\psi|^{-1}\end{aligned}\right\}$$

(Vertical motion Vertical shear term)

$$-\rho\frac{\partial V}{\partial s}\, V\frac{\partial\psi}{\partial n}|\nabla\psi|^{-1}$$

(Inflow feature term)

$$+\rho V\frac{\partial\psi}{\partial s}\, V\frac{\partial\psi}{\partial s}\,|\nabla\psi|^{-1}$$

(Rotational feature term)

$$-\rho K\, V\frac{\partial\psi}{\partial n}\,|\nabla\psi|^{-1}$$

(Frictional term)

$$+\rho f\, V\frac{\partial\psi}{\partial s}\,|\nabla\psi|^{-1}$$

(Coriolis term) (4)

Inflow feature term + Rotational feature term can be expressed as :

$$-\rho(\nabla.\mathbf{V})\, V\frac{\partial\psi}{\partial n}\,|\nabla\psi|^{-1}+\rho V^2|\nabla\psi|$$

$$\text{or } -\rho L_1|\nabla\psi|^{-1}+\rho\zeta V\frac{\partial\psi}{\partial s}\,|\nabla\psi|^{-1}$$

Making use of the rule in 3.5 we can state :

Friction — Friction associated with confluence favours a pressure minimum and with difluence, it favours a pressure maximum.

Coriolis parameter -- Counterclockwise/clockwise curvature in northern /southern hemisphere favours a pressure minimum and opposite curvatures favour pressure maximum.

Inflow/Outflow — Inflow and outflow feature favour a pressure maximum.

Rotational features – Counterclockwise and clockwise curvature features favour pressure minimum.

Local changes — Increase of speed at all point indicates intensification and decrease weakening. Hence intensification associated with confluence favours pressure minimum and weakening a pressure maximum. In the case of difluence just the opposite.

If the centre moves, $\partial\psi/\partial t$ is positive on one side and negative on the other. Hence second part of local change term in Eqn. (4) favours an isobar to pass through the centre.

Vertical motion with vertical shear — $\frac{\partial \log V}{\partial z}$ indicates change of speed with height. The intensity of cyclone decreases with height in which case this term is negative. There can be cases when this term is positive. Vertical motion can be positive as well as negative.

TABLE 3

w	$\frac{\partial \log V}{\partial z}$	$V\frac{\partial\psi}{\partial n}$	$-w\frac{\partial \log V}{\partial z}\,V\frac{\partial\psi}{\partial n}$	Pressure
+	—	—	—	Max.
—	—	—	+	Min.
+	+	—	+	Min.
—	+	—	—	Max.

The role of first part of vertical motion/vertical shear term in Eqn. (4) is given in Table 3 for cyclones.

A similar table for difluence can be constructed. If the axis of cyclone is tilted, $\partial\psi/\partial z$ is positive on one side and negative on the other. In such a case, the term $\rho w \frac{\partial\psi}{\partial z} V \frac{\partial\psi}{\partial s} |\nabla\psi|^{-1}$ favours an isobar to pass through the centre.

Now we derive expressions for vertical motion on the basis of ideas developed in previous chapter.

5.2. *Closed* $(\nabla\psi)$ *vector line*

We have already noted that $\oint(\nabla p).(d\mathbf{l})=0$. In the case of cyclone, $\nabla\psi$ vector lines enclose the centre. Let $\nabla\psi=|\nabla\psi|\, d\mathbf{l}$ and hence,

$$\oint\frac{(\nabla p).(\nabla\psi)}{|\nabla\psi|}\,dl=0$$

Replace ∇p from equation of motion and obtain :

$$\bar{w}=\frac{\oint\left\{\left(f+\frac{\partial\psi}{\partial t}\right)V\frac{\partial\psi}{\partial n}+\left(K+\nabla.\mathbf{V}+\frac{\partial}{\partial t}\log V\right)V\frac{\partial\psi}{\partial s}\right\}\rho|\nabla\psi|^{-1}\,dl}{-\oint\left(\frac{\partial\psi}{\partial z}V\frac{\partial\psi}{\partial n}+\frac{\partial}{\partial z}\log V\; V\frac{\partial\psi}{\partial s}\right)\rho\,|\nabla\psi|^{-1}\,dl} \tag{5}$$

THEOREM 6

Given a cyclonic wind field satisfying

$$\rho\left(\frac{d\mathbf{V}}{dt}+(\mathbf{k}\times\mathbf{V})\,f+K\mathbf{V}\right)=-\nabla p$$

such that streamlines are confluenting, *i.e.*, $\partial\psi/\partial n$ is negative and their curvature is

where $\bar{w}$ is the mean of w on the closed $\nabla\psi$ vector line.

5.2.1. *Orthogonality of isogon and isobar* — We noted that isogons and isobars tend to become nearly orthogonal when the pressure system becomes deep. If so, $(\nabla p)\,.\,(\nabla\psi)=0$

$$w=\frac{\left(f+\frac{\partial\psi}{\partial t}\right)V\frac{\partial\psi}{\partial n}+\left((K+\nabla\,.\,\mathbf{V}+\frac{\partial}{\partial t}\log V\right)V\frac{\partial\psi}{\partial s}}{-\left(\frac{\partial\psi}{\partial z}V\frac{\partial\psi}{\partial n}+\frac{\partial}{\partial z}\log V\;V\frac{\partial\psi}{\partial s}\right)} \tag{6}$$

counterclockwise/clockwise in northern/southern hemisphere, *i.e.,* $f(\partial\psi/\partial s)$ is positive and speed generally decreases with height, *i.e.,* $\partial(\log V)/\partial z$ is negative, then at the cyclone centre :

(*i*) friction, coriolis parameter, rotational feature, intensification and downward motion favour a pressure minimum,

(*ii*) inflow, weakening and upward motion favour pressure maximum and

(*iii*) motion of centre as well as tilt of axis favour an isobar.

Comment (*i*)

Ultimately the observed pressure field is due to all factors. Since rotation, friction and coriolis are predominant, pressure minimum is seen.

Comment (*ii*)

It is possible to have no closed isobar in case vertical shear associated with vertical motion

where orthogonal conditions are not exactly fulfilled, we have to add $-\rho^{-1}(\nabla p)\,.\,(\nabla\psi)$ on the numerator.

5.2.2. *Circular isobars*

A polar cooridinate system (r,θ) with origin coinciding with the centre of circular isobar is chosen and the equation of motion is split into radial and tangential component. The tangential component is :

$$\frac{dV_\theta}{dt}+fV_r+KV_\theta+\left(\frac{\partial\theta}{\partial t}+w\frac{\partial\theta}{\partial z}+\frac{V_\theta}{r}\right)V_r$$

$$=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}=0$$

where V_r and V_θ are radial and tagential components of wind vector. From this, we obtain:

$$w=\frac{\left\{f+\frac{\partial\theta}{\partial t}+V_r\frac{\partial}{\partial r}\left(\frac{V_\theta}{V_r}\right)\right\}V_r+\left(K+\nabla\,.\,\mathbf{V}+\frac{\partial}{\partial t}\log V_\theta\right)V_\theta}{-\left[\frac{\partial\theta}{\partial z}V_r+\frac{\partial}{\partial z}\log V_\theta\;V_\theta\right]} \tag{7}$$

and local change term are predominant and cyclone is in a very weak stage.

5. Vertical motion

5.1. Theoretical considerations require existence of $(\partial\mathbf{V}/\partial t)+w(\partial\mathbf{V}/\partial z)$ and observationally vertical motion and vertical shear are seen in cyclone.

If isobars are not exactly circular, and $\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}$ on the numerator. Note the remarkable similarities among the three expressions.

5.3. To gain an analytical insight, we propose using Lakshminarayanan's mathematical mode

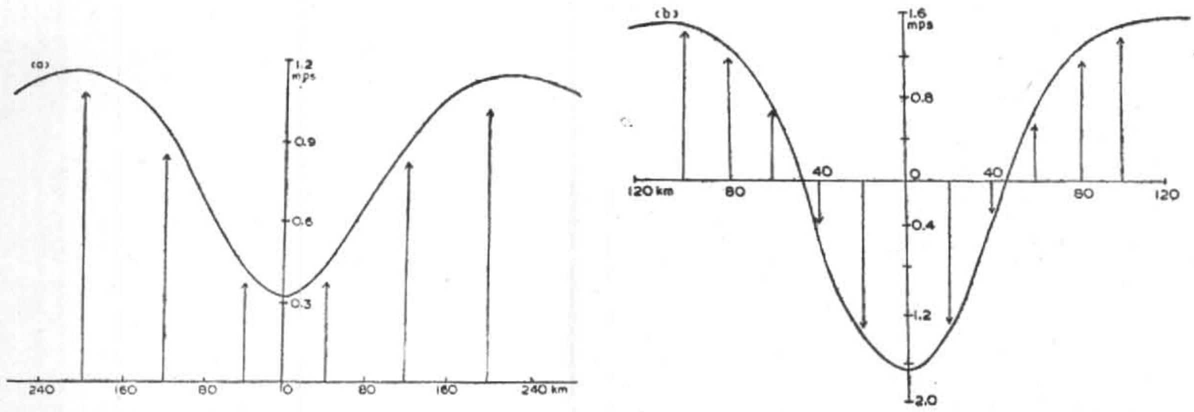


Fig. 5. East-west profile of vertical motion : (a) Case I — Weak stage, (b) Case II—Intense stage

as explained in 2.3 and evaluate vertical motion from Eqn. (6) :

$$w = \frac{\left[-\left(f + \frac{\partial\psi}{\partial t}\right)\frac{(V_i)_{max}}{(V_R)_{max}} + \left\{K + \frac{(V_i)_{max}\, e^{1/2}}{\sigma}\left(\frac{r^2}{\sigma^2} - 2\right)\phi + \frac{\partial}{\partial t}\log V\right\}\right]}{-\left(-\frac{\partial\psi}{\partial z}\,\frac{(V_i)_{max}}{(V_R)_{max}} + \frac{\partial}{\partial z}\log V\right)} \qquad (8)$$

For illustrative purposes, we assume following:

(*i*) axis is nearly vertical, *i. e*,. $\frac{\partial\psi}{\partial z} \simeq 0$

(*ii*) motion of the centre is small, *i. e.*,

$$\frac{\partial\psi}{\partial t} \simeq 0$$

(*iii*) Lat. 15° N; $f = .3775 \times 10^{-4}$ sec^{-1}

(*iv*) speed increases exponentially to double the value in 24 hours, *i.e.*,

$$\frac{\partial}{\partial t}\log V = \cdot 08 \times 10^{-4} \text{ sec}^{-1}$$

(*v*) speed decreases exponentially to half the value in 3 km, *i. e.*,

$$\frac{\partial}{\partial z}\log V = -2\cdot 31 \times 10^{-6} \text{ sec}^{-1}$$

(*vi*) $K = 2.5 \times 10^{-4}$ sec^{-1} (p. 661, *Compendium of Meteorology*)

Case **(*i*)** — *Weak cyclone/depression* : $(V_i)_{max}$ = 5 mps; $(V_R)_{max}$ = 15 mps; $(V)_{max}$ = 15·8 mps, σ=100 km. This is a weak stage of cyclone called depression.

TABLE 4

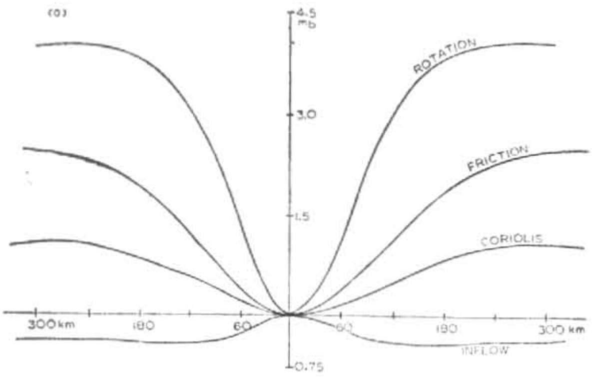
Vertical motion (mps)

Weak cyclone (Case I)		Intense cyclone (Case II)	
Distance (km)	Vertical motion (mps)	Distance (km)	Vertical motion (mps)
0	+0.34	0	—1.75
20	+0.37	10	—1.68
40	+0.45	20	—1.36
60	+0.56	30	—0.89
80	+0.68	40	—0.35
100	+0.81	50	+0.20
120	+0.95	60	+0.67
140	+1.04	70	+1.04
160	+1.09	80	+1.28
180	+1.13	90	+1.42
200	+1.14	100	+1.45
∞	+1.06	∞	+1.06

Case (*ii*) — *Intense cyclone* : $(V_i)_{max}$ = 10 mps; $(V_R)_{max}$ = 30 mps; $(V)_{max}$=31·6 mps, σ=50 km. This is an intense stage of cyclone. Compared to weak stage, speed has doubled and radius of ring of maximum wind has contracted to half the value in intense stage.

Case (*iii*)

Table 4 lists value of w in both the cases and Fig. 5(a & b) gives profile of w along east-west oriented radial distance.



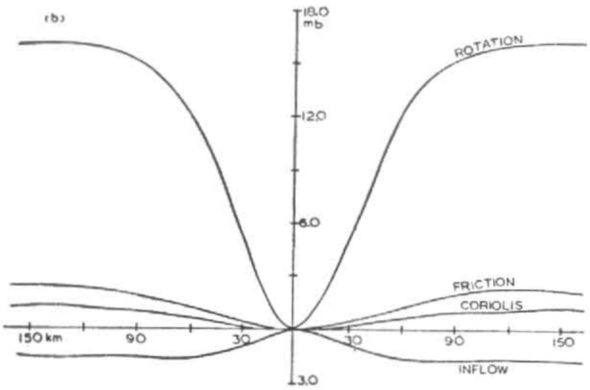


Fig. 6. East-west pressure profile : (a) Weak stage — Case I and (b) Intense stage — Case II

We note that :

(*i*) vertical motion is upward at all points with a minimum value at cyclone centre in the weak stage of cyclone.

(*ii*) As the cyclone intensifies characterised by increase of speed on the ring of maximum wind, increased inflow, increased rotational component and hence increased convergence and vorticity, vertical downward motion near cyclone centre occurs and elsewhere vertical upward motion continues. Vertical downward in the centre of cyclone is characterised by absence of cloud and is picturesquely referred to as eye of the cyclone.

We conclude that as the cyclone intensifies by increase of speed on RMW and/or contraction of RMW and hence increase of both convergence and vorticity, the eye forms where downward vertical motion and hence absence of clouds takes place. In reaching this conclusion, we require nearly vertical axis and small motion of the centre. Further, we make use of Theorem VII as in 3.4.

TABLE 5

Pressure field (in mb)

r/σ	Distance (km)	Inflow feature	Rotational feature	Friction	Coriolis
(a) Weak cyclone/depression — Case I					
.2	20	—0.031	+0.156	+0.054	+0.025
.6	60	—0.220	+1.200	+0.397	+0.179
1.0	100	—0.397	+2.509	+0.947	+0.429
1.4	140	—0.450	+3.410	+1.504	+0.681
1.8	180	—0.432	+3.814	+1.931	+0.875
2.2	220	—0.409	+3.938	+2.194	+0.994
2.6	260	—0.399	+3.965	+2.326	+1.053
3.0	300	—0.397	+3.696	+2.381	+1.079
(b) Intense stage — Case II					
.2	10	—0.123	+0.623	+0.054	+0.025
.6	30	—0.879	+4.802	+0.397	+0.179
1.0	50	—1.588	+10.037	+0.947	+0.429
1.4	70	—1.802	+13.641	+1.504	+0.681
1.8	90	—1.727	+15.258	+1.931	+0.875
2.2	110	—1.636	+15.754	+2.194	+0.994
2.6	130	—1.598	+15.861	+2.326	+1.053
3.0	150	—1.588	+15.878	+2.381	+1.079

6. Pressure field of cyclone

6.1. We take the same cyclone model which we used for obtaining vertical motion, to get pressure field :

$$(p-p_c)=\tfrac{1}{2}\rho(V_i)^2_{\max}\left[-e+\left(1-\frac{r^2}{\sigma^2}\right)\exp\left(1-\frac{r^2}{\sigma^2}\right)\right]$$

(Inflow feature)

$$+\tfrac{1}{2}\,\rho\,(V)^2_{\max}\left[\,e-\exp\left(1-\frac{r^2}{\sigma^2}\right)\right]$$

(Rotational feature)

$$+\rho K(V_i)_{\max}\,\sigma\left[e^{\frac{1}{2}}-\exp\left(\frac{1}{2}-\frac{r^2}{2\sigma^2}\right)\right]$$

(Frictional term)

$$+\,\rho f\,(V_R)_{\max}\,\sigma\left[e^{\frac{1}{2}}-\exp\left(\frac{1}{2}-\frac{r^2}{2\sigma^2}\right)\right]$$

(Coriolis term) (9)

6.2. Table 5 gives termwise pressure field contribution for weak stage (case I) and intense stage (case II). Since we have taken rotation feature as predominant, the contribution due to rotational feature is predominant, next is friction and coriolis parameter is least due to choice of Lat. 15°N. Inflow contribution is just the opposite to other terms as required by theory. Fig. 6 (a & b) gives pressure profile along east-west oriented radial line.

7. Discussion

7.1. Pressure and wind relationship is basic and fundamental not only in meteorology but also in hydro/fluid dynamics. Hence the available literature is extensive and vast on this topic. Well known relations like Bernoulli's theorem, geostrophic and gradient wind equations are none but solutions of Eqn. (1) under various constraints. The two basic difficulties in dealing with pressure field associated with a spiralling wind field are attributable to lack of :

(*i*) definition of spiral properties,

(*ii*) adequate mathematical model.

The properties of spiral are given in a set of six theorems by the author (Lakshminarayanan 1978) and a simple mathematical model for spiralling circulation has been formulated by him (Lakshminarayanan 1975). The set of theorems and the model are used in this paper.

7.2. A deliberate attempt is made to cast mathematical expressions to follow the inherent natural configurational geometry of isogons, isopleths of speed, streamlines, curvature, confluence/difluence etc so that conclusions are easily interpretable in terms of prominently recognisable cyclone features. Isogon coordinate system is especially chosen instead of cartesian/polar coordinate systems extensively used in meteorology. The initial difficulty in getting familiar with isogon coordinate system is adequately compensated by analytical ease/clarity of expressions in interpreting cyclone features. With a view to overcome difficulties, if any, we have listed in Table 1 components along and normal to isogons.

7.3. Geostrophic and gradient wind relations are popular in meteorology. In both the cases streamlines and isobars are the same. Inflow features of cyclones with closed isobars necessitate cross isobaric flow. We establish that inflow features of a cyclone favour a maximum pressure at cyclone centre. Friction with confluence, coriolis parameter with counterclockwise/clockwise curvature in northern/southern hemisphere, rotational feature, and growth with confluence favour a minimum pressure at cyclone centre. Inflow and decay with confluence favour a maximum presssure. The role of vertical motion associated with vertical shear and confluence favouring a minimum/maximum is given in the text of this paper. Movement of centre and tilt of axis favour an isobar. All other terms put together favour a constant pressurefield or added together must vanish as exemplified by :

$$(\nabla p).(\nabla\psi)=0 \quad \text{or} \quad \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}=0$$

In the atmospheric context, rotational feature, friction and coriolis parameter in that order contribute to low pressure with a minimum.

7.4. Since vertical motion in atmosphere is relatively smaller compared to horizontal motion, the equation of motion for horizontal wind vector is not customarily used to evaluate w. Instead equation of continuity is used. In this paper, vertical motion is obtained from the equation of motion itself. The eye of cyclone where vertical downward motion can be inferred to take place is shown to occur when the cyclone is intense.

8. Conclusion

The role of inflow, rotation, friction, coriolis parameter, local change and vertical motion associated with vertical shear in forming the pressure field of a cyclone is investigated. Formulation of the eye is shown to occur when the cyclone is intense.

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