

A mathematical model for QPF for flood forecasting purposes

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सार — भौतिक और गतिकीय नियमों के आधार पर मातात्मक वर्षण पूर्वानुमान के लिए एक गणितीय निदर्श का विकास किया गया। सतही और ऊपरी वायु के मौसम विज्ञान संबंधी प्रेक्षणों का प्रयोग निदर्श (मॉडल) में निवेश के रूप में किया गया। निर्गम वर्षण की वह दर है जिसमें से वर्षण की राशि का आकलन समय समाकलन पर किया जा सकता है। इस निदर्श का प्रयोग परिचालन रूप में वर्षा के पूर्वानुमान के लिए किया जा सकता है।

ABSTRACT. A mathematical model for Quantitative Precipitation Forecasting (QPF) has been developed on the basis of physical and dynamical laws. The surface and upper air meteorological observations have been used as inputs in the model. The output is the rate of precipitation from which the amount of precipitation can be computed on time integration. The model can be used operationally for rainfall forecasting.

Key words — Flood forecasting, mathematical model, precipitation efficiency, precipitation rate, quantitative precipitation forecast.

1. Introduction

Attempts have been made in several countries to forecast the amount of precipitation in different catchment areas with different lead time (the difference between the time of the occurrence of the forecasted phenomenon and the time when the forecast is issued) using various techniques but have achieved only limited success. A very good account on the status of Quantitative Precipitation Forecasting (QPF) models used in various countries for operational purposes has been described by Bellocq (1980). The recent advances in QPF research and possible future directions towards achieving improved use of QPF information in hydrological forecasting have been discussed at length by Georgakakos and Hudlow (1984).

The major rain producing system in the Indian sub-continent is the southwest monsoon or Asiatic summer monsoon (June to September) which is typical of this part of the world. Some parts or other in India come under the grip of flood due to heavy rainfall in the catchment areas during the southwest monsoon period every year causing loss of lives and damage to standing crops and property. India Meteorological Department has been issuing semi-QPF for various river catchments in India from the Flood Meteorological Offices (FMO) located at various locations using different techniques (*viz.*, synoptic, statistical, synoptic statistical) for more than a decade. Attempts are also being made to develop mathematical model for issuing QPF in India for operational purposes.

In this paper an attempt has been made to develop a mathematical model for QPF, based on physical and dynamical laws, which could be used for operational

purposes. The inputs are the surface and upper air meteorological observations and the output is the rate of precipitation from which amount of precipitation can be computed readily on time integration. This forecast amount of rainfall can again be used as an input in hydrological model for flood forecasting purposes.

2. Formulation and description of the model

The atmospheric model is designed to recognise the presence of water either as vapour or liquid. The concentration of water vapour may be represented by the specific humidity q , defined as the ratio of the mass of water vapour to the mass of moist air.

The present mathematical model is based on the hypothesis that the specific humidity in an air column is conserved. In other words, it is based on the principle of conservation of specific humidity. In mathematical notation it can be expressed as :

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} + \omega \frac{\partial q}{\partial p} = 0 \quad (1)$$

$$\text{where, } u = \frac{dx}{dt}, v = \frac{dy}{dt} \text{ and } \omega = \frac{dp}{dt}$$

The precipitable water vapour in a column of air is the total mass, R_F of water vapour per unit area in the column. Symbolically (Haltiner and Martin 1957)

$$R_F = \int_{z_s}^{z_t} \rho_v \delta z \quad (2)$$

where, ρ_v is the density of water vapour and z_s and z_t the elevation of the bottom and the top of the air column respectively.

TABLE 1
Precipitation rate for New Delhi

Date (1984)	Forecast of rainfall rate for the next 24 hours (mm/hr)	Actual rate of precipitation (mm/hr)
1 Jul	0.94	0.91
2 Jul	0.62	0.85
3 Jul	0.25	0.06
4 Jul	0.00	Trace*
25 Aug	0.40	0.62
26 Aug	2.04	1.71
27 Aug	0.70	1.96
28 Aug	0.08	0.08
29 Aug	0.60	Trace*

*Trace means rainfall of 0.4 mm or less in 24 hours.

TABLE 2
Precipitation rate for Bombay

Date (1989)	Forecast of rainfall rate for the next 24 hours (mm/hr)	Actual rate of Precipitation (mm/hr)	
		Colaba	Santacruz
20 Jul	0.60	0.06	0.30
21 Jul	3.02	3.23	3.92
22 Jul	1.75	2.25	2.91
23 Jul†	1.03	5.40	4.38
24 Jul†	1.61	1.01	2.41
25 Jul	0.42	0.64	0.82
26 Jul	0.58	0.36	0.53

†Based on Radiosonde data of 1200 UTC because 00UTC flight terminated in the lower troposphere and the actual precipitation rate is calculated from the rainfall between 1200 UTC and 2400 UTC.

The Eqn. (2) can be written in the isobaric coordinates as :

$$R_F = -\frac{1}{g} \int_{p_s}^{p_t} q \delta p \quad (3)$$

where, $q = \rho_v/\rho =$ Specific humidity.

Since q is a non-dimensional quantity, the variable R_F has the dimensions of mass per unit area. R_F may be converted to the parameter 'precipitable water' by dividing R_F by the density of water. We can now derive an expression for the rate of precipitation, $\partial R/\partial t$ from Eqn. (3) in the following way :

$$\frac{\partial R}{\partial t} = E \frac{\partial R_F}{\partial t} \quad (4)$$

where, E ($0 \leq E \leq 1$) is a multiplication factor. The multiplication factor has been used because R_F is the amount of precipitable water vapour not the actual amount of precipitation. $R_F = R$, if and only if all the available moisture condenses and falls as precipitation. But in the actual atmosphere it has been found that only a part of the precipitable water vapour gets converted into precipitation. Thus E is a measure of the proportion of the available moisture which precipitates and may be termed as precipitation efficiency. This is a key parameter which takes care of moisture loss due to

evaporation and other aspects of cloud microphysics.

Combining Eqns. (3) and (4) we get :

$$g \frac{\partial R}{\partial t} = gE \frac{\partial R_F}{\partial t} = -E \frac{\partial}{\partial t} \left\{ \int_{p_s}^{p_t} q \delta p \right\} \\ = -E \left\{ \int_{p_s}^{p_t} \frac{\partial q}{\partial t} \delta p \right\} \quad (5)$$

It has been assumed in the above that the variation of p_t and p_s with time counter balance each other.

Using Eqn. (1) we get from Eqn. (5) :

$$g \frac{\partial R}{\partial t} = E \int_{p_s}^{p_t} \left(u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} \right) \delta p + E \int_{p_s}^{p_t} \omega \frac{\partial q}{\partial p} \delta p \\ \text{or, } g \frac{\partial R}{\partial t} = E \sum_{i=1}^n (p_{i-1} - p_i) \left\langle u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} \right\rangle_{i, i+1} \\ + E \sum_{i=1}^n (q_{i+1} - q_i) \left\langle \omega \right\rangle_{i, i+1} \quad (6)$$

$i=1$, corresponds to the surface pressure p_s and $i=n$ to the pressure surface p_t and $\left\langle \right\rangle_{i, i+1}$ represents average value of the layer bounded by the surfaces i and $i+1$.

The first term on the right hand side of the Eqn. (6) actually represents the horizontal advection of moisture and may be termed as the advection term and the second term represents the effect of the vertical velocity on moisture and may be termed as the vertical velocity term. Thus Eqn. (6) shows that the velocity field especially the vertical velocity coupled with information on humidity, is related to the rate of precipitation in a large area.

Precipitation is the end product of the physical processes taking place in the atmosphere. The occurrence of precipitation is strongly controlled by the motion of the cloud air. In other words, rainfall is always associated with clouds and the moisture advection normally takes place at the boundary layer which almost coincides with the base of the clouds. It would be quite reasonable if 850 hPa level is chosen as the top of the boundary layer. In that case the summation in the first term on the right hand side of Eqn. (6) need not be performed up to the top of the air column; the summation up to the top of the boundary layer would be sufficient. Thus, the Eqn. (6) may be modified as :

$$g \frac{\partial R}{\partial t} = E (\Delta p)_{p_s, 850} \left\langle v \cdot \nabla q \right\rangle_{p_s, 850} \\ + E \sum_{i=1}^n (\Delta q)_{i, i+1} \left\langle \omega \right\rangle_{i, i+1} \quad (7)$$

where, $(\Delta p)_{p_s, 850} = 850 - p_s$; $(\Delta q)_{i, i+1} = q_{i+1} - q_i$.

Since the parameter q , the specific humidity, is not directly measured, it is desirable that it be expressed in terms of some parameters, those are either measured at the observational sites or at least reported in the synoptic or upper air observations.

TABLE 3
Precipitation rate for Ahmedabad

Date (1989)	Forecast of rainfall rate for the next 24 hours (mm/hr)	Actual rate of precipitation (mm/hr)
23 Jul	1.10	2.10
24 Jul	ND	1.41
25 Jul	0.90	0.34
26 Jul	1.50**	0.25
27 Jul	0.00	0.02
28 Jul	0.80	0.52
29 Jul	0.02	0.01

ND—No Data;

**Radiosonde data available only up to 700 hPa.

By definition, the specific humidity is given by :

$$q = \frac{\rho_v}{\rho} = \frac{\rho_v}{\rho_d + \rho_v} \quad \text{where, } \rho_d = \text{Density of the dry air}$$

$$q = \frac{0.622 e_s(T_d)}{p - 0.378 e_s(T_d)} \quad (8)$$

where, $e_s(T_d)$ is the saturation vapour pressure over a plane surface of pure water; T_d , the dew point temperature. The saturation vapour pressure is a nonlinear convex function of temperature. A convenient formulation for determining the saturation vapour pressure $e_s(T_d)$ in terms of T_d may be used for this purpose. The polynomial relation suggested by Lowe and Ficks (1974) has been found to provide an excellent fit with the observed ones in the range -50°C to $+50^\circ\text{C}$ (Pruppacher and Klett 1980). The formulation reads as :

$$e_s(T_d) = \sum_{n=0}^6 a_n T_d^n, \quad \text{with } T_d \text{ (}^\circ\text{C)} \text{ and } e_s \text{ (hPa)} \quad (9)$$

where,

$$\begin{aligned} a_0 &= 6.107799961, & a_4 &= 3.031240396 \times 10^{-6} \\ a_1 &= 4.436518521 \times 10^{-1}, & a_5 &= 2.034080948 \times 10^{-8} \\ a_2 &= 1.428945805 \times 10^{-2}, & a_6 &= 6.136820929 \times 10^{-11} \\ a_3 &= 2.650648471 \times 10^{-4} \end{aligned}$$

Using Eqns. (7), (8) and (9) the rate of precipitation may be computed provided the value of E is available. Normally the rate of precipitation is expressed in $\text{kg m}^{-2} \text{sec}^{-1}$. But since 1 kg m^{-2} of liquid water is equivalent to the depth of 1 mm of rainfall, $\partial R/\partial t$ can be expressed directly as mm sec^{-1} . The vertical p -velocity ω may be computed using either the equation of continuity in isobaric co-ordinates or the diagnostic ω -equation. The second method is more accurate.

The method suggested by Sulakvelidze (1969) and applied by Georgakakos and Bras (1984a and b) for vertical velocity can also be used. The expression for the vertical velocity used by them is

$$w = a \sqrt{c_p \Delta T} \quad (10)$$

where, a is a constant parameter, $\Delta T = |T_m - T_s|$, T_m is the parcel temperature ($^\circ\text{K}$) at a certain level p (hPa) assuming pseudoadiabatic ascent and T_s is the corresponding ambient air temperature ($^\circ\text{K}$). The square of the quantity ' a ' is analogous to the ratio of the kinetic to the thermal energy per unit mass of ascending air, at the level p . Therefore, ' a ' is a non-dimensional quantity.

TABLE 4
Precipitation rate for Nagpur

Date (1989)	Forecast of rainfall rate for the next 24 hours (mm/hr)	Actual rate of precipitation (mm/hr)
22 Aug	0.80	0.02
23 Aug	0.30	0.11
24 Aug	0.60	2.19
25 Aug	0.90	0.20
26 Aug	0.30	0.23
27 Aug	0.20	0.24

When the parcel temperature is more than the environmental temperature there would be upward vertical motion and opposite is the case when the parcel is cooler than the environment. Georgakakos and Bras (1984a and b) have found out that a value of 0.002 for ' a ' gave a good fit. From the Eqn. (10) we can compute the vertical p -velocity ω using the following relation (Holton 1979) :

$$\omega = -g\rho w \quad (11)$$

3. Estimation of precipitation efficiency

The quantity precipitation efficiency E , is a complex, elusive factor to determine quantitatively. It is not theoretically spatially constant either. Rhea (1978) used precipitation efficiency (which is slightly different from the precipitation efficiency defined here) is his orographic precipitation model and reported that many computations of precipitation rates for hydrometeorological purposes routinely set $E=1$ (i.e., they equate condensation supply rate of precipitation rate). A large number of test cases were run by Rhea using a variety of E values and it was found that on 50% occasions a value of 0.25 for E and at least 70% occasions a value of 0.21 or greater gave a good fit with the observed values. Moreover, it has been estimated that only about 30% of the moisture falls out as precipitation (Haltiner and Martin 1957). Thus it would be reasonable if a value of 0.20 to 0.30 for E is chosen considering the loss due to evaporation of the droplets of precipitation. But it is recommended that the value of E be estimated for each individual station with a long series of data with varieties of rain storms. However, it is to be remembered that extreme parameter sensitivity is not desirable considering the crudeness of the input data as well as the precipitation measurement.

4. Case studies

To test the model, computation of rainfall rate has been done for New Delhi during the southwest monsoon of 1984 and for Bombay, Ahmedabad and Nagpur for 1989. Upper air data for New Delhi, Bombay, Ahmedabad and Nagpur up to 300 hPa level and the upper air data along with wind observations for neighbouring stations up to 850 hPa levels for 00 UTC have been utilized.

The vertical velocity has been computed using the scheme of Sulakvelidze (1969) and Georgakakos and Bras (1984a & b) and the Eqn. (11) of the previous section. The results have been presented in the Tables 1, 2, 3 and 4. In this computation the efficiency parameter E has been taken as 0.30.

It can be seen from the above that the model forecasts for rainfall rate are in reasonable agreement with the

actual values. In some cases, of course, the difference is quite large. However, the correlation between the model forecast and the actual has been found to be 0.60. In the above the vertical p -velocity has been computed using Sulakvelidze scheme. In this scheme the vertical velocity is a function of stability. In other words, if the atmosphere is unstable the vertical velocity will be upward. In case of stratiform cloud which is seen during the southwest monsoon period the vertical velocity computed by the above scheme would be lower than the actual vertical velocity. Thus the model will have a tendency to underestimate the rainfall from stratiform cloud. In absence of diagnostic Omega values the Sulakvelidze method is, of course, most suitable.

5. Discussion

The model described in this paper is very simple and can be used for operational purposes. The model is based on the hypothesis that the specific humidity of the atmospheric column under consideration is conserved. This assumption is valid as long as there is no change in phase which means once the process of condensation starts the specific humidity may not remain constant. Since in this model we are calculating the rate of change of precipitable water vapour in the atmospheric column and the rate of precipitation is computed from it through a factor called precipitation efficiency, the above assumption may be considered valid. The same hypothesis has been taken into consideration in the limited area fine mesh model (LFM) in the National Weather Service (NWS) of the United States of America (Gerrity 1977; Newell and Deaven 1981). Several workers in USSR attempted to develop a method to forecast cloudiness and precipitation on the basis of the same hypothesis (for complete list see Matveev 1967a).

This model may be reasonably successful for predicting precipitation associated with large-scale (synoptic scale) disturbances and may not be successful to that extent for the prediction of orographic and convective precipitation. The orographic vertical motion is frequently an order of magnitude larger than that associated with large-scale vertical velocity, while the vertical motion associated with meso-scale convective system may be still higher. Both orographic and convective element vertical motions are small scale phenomena, implying action on a given air parcel for only a short time whereas the large-scale vertical motion field slowly displaces a given parcel for an extended period. Thus each may have a considerable influence on the total precipitation process.

Matveev (1968b) has reported an interesting phenomenon associated with cumulonimbus (*Cb*) cloud. He has reported from the data of 26 cases in the Kiev region in Soviet Union that the precipitation exceeded the cloud precipitable water content on the average by a factor of 8.8 (with variations between 1.8 and 16.9). The water reserves of a *Cb* cloud are replenished every 7-12 minutes. The above data show that the amount of precipitation from the *Cb* cloud systems during their time of existence exceeds by about one order of magnitude of their water content at any given moment. This means that the water is completely renewed many times during the cloud's existence.

The above observation confirms the fact that the forecasting of the amount of rainfall associated with the convective cloud is a very difficult proposition if not impossible. As such QPF itself, for any type of precipitation is a very difficult task.

Moreover, unlike other meteorological parameters rainfall is a highly variable quantity. Precipitation amounts are seldom representative and a few rain gauges do not constitute an adequate sample of a large area for quantitative purposes. Moreover, there are several methods for determining the average depth of precipitation and the amount computed by one method differs considerably from that computed by another method.

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