Mausam, (1988), **39**, 1, 37-44

551.515

Path of a vortex in a semi-infinite region with a streaming motion along a plane wall, past a circular island at a finite distance from the wall

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(*Received 22 December 1984*)

सार — इस शोधपत्र में एक वृत्तीय द्वीप से सीमित दूरी की दीवार और दीवार के सामानान्तर असीमित दूरी पर सतत-समान प्रवाह के साथ सीधी दीवार से घिरे हुए अर्ध–असीमित क्षेत्र में भ्रमिल के पथ का अध्ययन करने का प्रयास किया गया है। इससे एक समुचित रूपान्तरण निकाला गया है। जिसकी सहायता से प्रवाह और मूल–स्तर में भ्रमिल–प्रणाली के बिम्ब की स्थितियां प्राप्त की गई हैं। यह पाया गया है कि द्वीप के केन्द्र से गुजरने वाली दीवार से लम्ब रेखा पर स्थाई बिन्दु विद्यमान है। इनमें से एक समान प्रवाह की शक्ति के धनात्मक अनुपातों के मामले में द्वीप और दीवार तथा भ्रमिल के बीच होता है और ऋणात्मक अनुपातों के मामले में एक समतल दीवार और द्वीप में तथा दूसरा द्वीप से परे होता है।

ABSTRACT. An attempt has been made to study the path of a vortex in a semi-infinite region bounded by a straight wall with steady uniform stream at infinity parallel to the wall and past a circular island at a finite distance from the wall. A suitable transformation has been found out with the help of which the flow itself and the positions of the image system of the vortex in the original plane have been obtained. It has been found that stationary points exist on the line perpendicular to the wall through the centre of the island — one between the wall and the island for positive ratios of the strength of the uniform stream and vortex and two, in case of negative ratios — one between the plane wall and island and other away from the island.

1. Introduction

Increasing awareness of and interest in the orographical effects on the atmospheric flow in recent years have drawn attention for lead to detailed studies in this field. It has been recognised that mountain ranges do have strong interaction with and influence over the path and speed of movement of the atmospheric vortex like a storm or low pressure system, when it is in its vicinity. In studying the surface wind fields and tracks of typhoons, when encountering the island of Taiwan using field data, Chu *et al.* (1977) remarked that the phenomena associated with the interactions of typhoon with mountain barrier are quite complex. A series of laboratory experiments have been carried out by Pao (1976) to study the mechanical encounter of a vortex with two dimensional elliptical barrier and demonstrated that many distinctive flow characteristics associated with terrain effects, such as typhoon encountering the island of Taiwan can reasonably be simulated in the laboratory. Chang and Chen (1969) and Chang *et al.* (1975) have made some experimental simulation studies on the structure and topographical influence on typhoon. In a previous paper, the author (Mandal 1980) discussed the motion of a vortex in the presence of a straight boundary with semi-circular bay or bulge.

To get some insight, as to how the motion of an atmospheric vortex is influenced by the presence of an isolated hill near a mountain range or a high island near a high coast; a theoretical attempt has been made in this paper to study the motion of a vortex in two dimensions in the presence of two boundaries, *viz.*, a straight long wall and a circular island at a finite distance from it, with streaming motion parallel to the wall and uniform at a large distance from the island. The mathematical treatment in such a case is complicated. A technique has been developed to obtain the solution for uniform stream and a series solution has been obtainned. Further, one boundary is curved and separated from another with fluid; it is not possible to replace these two boundaries by a single straight boundary to make the study of the motion of the vortex simple. In this study, the positions and strengths of the image vortices in the original plane, corresponding to the infinite number of image vortices in transformed plane with two parallel boundaries have been derived and a series solution obtained; which have been added to get a closed form of the solution.

Another example of the barrier as studied in this work is an island near the bank of a broad river. It is also applicable for the flow of water in river or bay in the presence of island excluding the vortex part from solution.

2. Transformation

In Fig. 1, A_∞ B_∞, is an infinite, straight and rigid boundary wall. A circular island of radius a is at a

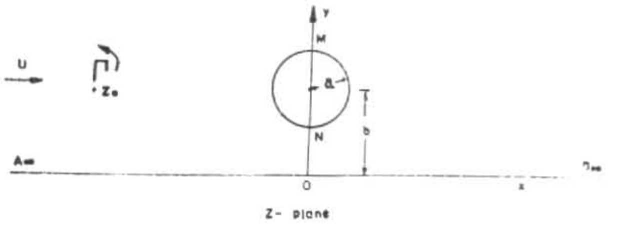

Fig. 1. Position of island with respect to plane wall boundary

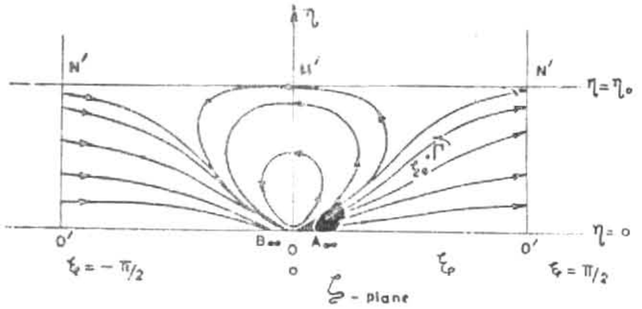

Fig. 2. Straight wall and boundary island in transformed plane

distance b from the wall and the fluid is confined to the upper infinite region outside the island.

We take the plane of motion as z $(=x+iy)$-plane. The axis of x, is taken along the boundary of the wall and axis of y, perpendicular to it through the centre of the island.

The semi-infinite region outside the island is occupied by an inviscid fluid which has at a large distance from the island a steady uniform flow U, parallel to x-axis. Let a vortex of strength, Γ be placed at a point, z_0 $(=x_0+iy_0)$. U, is considered positive for flow from left to right and Γ, is positive for cyclonic flow and negative for anticyclonic.

As the semi-infinite region outside the island cannot be transformed into a semi-infinite region, because the rigid boundaries are separated from each other by a region occupied by fluid, usual standard procedure cannot be used.

In order to confine the fluid regime within a region bounded by straight boundaries we will transform the region of interest from z-plane to ζ $(\xi+i\eta)$-plane (Fig. 2) using the relation :

$$z=-g\cot\zeta \tag{1}$$

where g is a constant parameter, which is related to the radius of the island a and its distance from the wall b; obviously b is greater than a.

In terms of real and imaginary parts:

$$\frac{x}{g}=-\frac{\sin 2\xi}{\cosh 2\eta-\cos 2\xi}$$

$$\frac{y}{g}=\frac{\sinh 2\eta}{\cosh 2\eta-\cos 2\xi} \tag{2}$$

After simplification

$$x^2+(y-g\coth 2\eta)^2=g^2\operatorname{cosech}^2 2\eta \tag{3}$$

Similarly, $(x\sin 2\xi+g\cos 2\xi)^2+y^2\sin^2 2\xi=g^2$

when, $\eta=0$; $y=0$

when, $\xi=\pm\pi/2$; $x=0$.

For, $\eta=\eta_0$, constant (not zero), the curves in the z-plane are circles with centre given by $(0, g\coth 2\eta_0)$ and radius $g\operatorname{cosech} 2\eta_0$.

In order to get the set of values of $2\eta_0$ and g for a particular set of values of a and b, if we put $a=g\operatorname{cosech} 2\eta_0$ and $b=g\coth 2\eta_0$, we get the following relations :

$$\left.\begin{aligned}\cosh 2\eta_0&=b/a\\ g^2&=b^2-a^2\end{aligned}\right\} \tag{4}$$

Thus, for known a and b $(b>a)$, the values of η_0 and g are uniquely determined.

As, $\eta=0$, when $y=0$, in the and ζ-plane, $\eta=0$ and $\eta=\eta_0$ from $\xi=-\pi/2$ to $\xi=\pi/2$

represent the straight and circular boundaries respectively of the z-plane . The entire fluid region is thus confined within the rectangle $\xi=\pm\pi/2$ and $\eta=0$ & η_0.

This transformation is equivalent to a cut along the imaginary axis from the lowest point of the island to the origin in z-plane, a counter clockwise rotation of angle π, open out of the island and coincidence of all points at infinity of z-plane at the origin in ζ-plane. Fig. 2 shows the transformation.

3. Complex potential for the motion of the fluid

3.1. *Due to the uniform flow*

The uniform flow U, at infinity in z-plane, has been reduced to a doublet at the origin in ζ-plane and due to periodic nature of the transformation function, the doublet will be repeated within the infinite strip $\eta=0$ and $\eta=\eta_0$; and hence the streamlines of the original doublet at $\zeta=0$, will cut the lines $\xi=\pm\pi/2$ orthogonally.

Since, $\eta=0$ and $\eta=\eta_0$, represent rigid boundaries' these can be considered to be removed without affecting the flow pattern by considering the image system with respect to these boundaries. The infinite numbers of images of the parent doublet will lie at points :

$$\xi=0 \tag{5}$$

$$\eta=\pm 2n\eta_0$$

where, $n=1,2,3,\ldots\ldots,\infty$

and all will have the same strength, which depends upon the strength U in z-plane. Thus in the entire ζ-plane, there will be infinite number of columns of infinite doublets; but one column on the imaginary axis is sufficient to study the flow. Now, in the reverse way, if one column of doublets in ζ-plane be transformed to the corresponding points in z-plane (of course, the strength will vary in that plane according to their positions), it will give the motion of the fluid in z-plane due to uniform stream at infinity past a circular island placed at a finite distance from the plane boundary.

It can be shown from Eqn. (2) that all the doublets in z-plane will lie on the imaginary axis and their positions are given by :

$$y_n = g \coth(\pm 2 n \eta_0) \qquad (6)$$

where, $n = 1, 2, 3, \ldots\ldots\ldots, \infty$

To find the strength μ, of the doublet in ζ-plane, we will use the condition that the complex potential $w = \mu/\zeta$ in ζ-plane, is equivalent to $w = Uz$ in z-plane as $z \to \infty$ ($\zeta \to 0$). We can write,

$$\frac{dw}{d\zeta} = \frac{dw}{dz} \times \frac{dz}{d\zeta}$$

since, $\dfrac{dw}{d\zeta} = -\mu/\zeta^2$, $\dfrac{dz}{d\zeta} = g/\sin^2\zeta$ and

$$\frac{dw}{dz} = -U \text{ as } \zeta \to 0$$

$$\mu = \lim_{\zeta \to 0} Ug\,\zeta^2/\sin^2\zeta = Ug \qquad (7)$$

The strengths of the image doublets at the corresponding points in z-plane will vary and depend on their positions in that plane and are given by :

$$\mu_n = \mu \left| \frac{dz}{d\zeta} \right| = Ug^2 \operatorname{cosech}^2 2n\eta_0 \qquad (8)$$

where, $n = \pm 1, \pm 2, \pm 3, \ldots\ldots\ldots\ldots\ldots, \pm\infty$.

The original doublet in ζ-plane is but the uniform stream at infinity in z-plane, and the virtual positions of the image doublets are on the imaginary axis, inside the island and outside the plane wall.

Therefore, the complex potential W_1, for the motion of the fluid in presence of a circular island of radius a and centre at a distance b from the plane boundary is given by the following expression by considering the virtual doublets in addition to the uniform stream U at infinity:

$$W_1 = -Uz - 2Ug^2 z \sum_{n=1}^{\infty} \frac{\operatorname{cosech}^2 2n\eta_0}{z^2 + g^2 \coth^2 2n\eta_0} \qquad (9)$$

where, $g^2 = b^2 - a^2$

$$2\eta_0 = \tfrac{1}{2} \ln \frac{b+g}{b-g}, \text{ and } \eta_0 \neq 0$$

At large distances all terms on the r.h.s. of Eqn. (9) except the first, *i.e.*, $-Uz$, vanish and the complex potential reduces to $W_1 = -Uz$, which is the simple case of uniform flow parallel to the real axis in the absence of the island.

3.2. *Due to vortex*

The strength of the vortex due to transformation is not changed. As in the case of doublet, the rigid boundaries in the ζ-plane can be considered as removed by considering a column of infinite number of images of the vortex with respect to the boundaries. Though, this column will be repeated due to the periodic nature of the transformation, it will be sufficient to consider one column only.

In the reverse way, if image vortices are virtually placed at the corresponding points in z-plane in addition to the original one, the system together will give the motion of the fluid due to the vortex in presence of the island and straight boundary in absence of any other motion.

The infinite number of images of the vortex can be grouped into two categories, *viz.*, positive and negative. The first negative image will be at the complex conjugate point of the orginal vortex.

If, ζ_0 be the position of the vortex corresponding to z_0, in z-plane , then the positions of the positive image vortices including the original one are given by :

$$\zeta_m = \zeta_0 \pm 2mi\eta_0 \qquad (10a)$$

and negative images by :

$$\zeta_{-m} = \bar{\zeta}_0 \pm 2mi\eta_0 \qquad (10b)$$

where, $m = 0, 1, 2 \ldots\ldots\ldots, \infty$

and $\bar{\zeta}_0$, is complex conjugate of ζ_0.

Here, like ζ_m corresponds to z_m, and ζ_{-m} to z_{-m} where, $m = 0, 1, 2, 3, \ldots\ldots\ldots, \infty$. The complex potential W_3, in the ζ-plane due to vortex at ζ_0 and all positive images are given by :

$$W_3 = i\Gamma \sum_{m=0}^{\infty} \ln(\zeta - \zeta_m)$$

and complex potential W_4 for negative images by :

$$W_4 = -i\Gamma \sum_{m=1}^{\infty} \ln(\zeta - \zeta_{-m})$$

Putting the actual values of ζ_m and ζ_{-m}, adding the two infinite series together and with some simplifica-

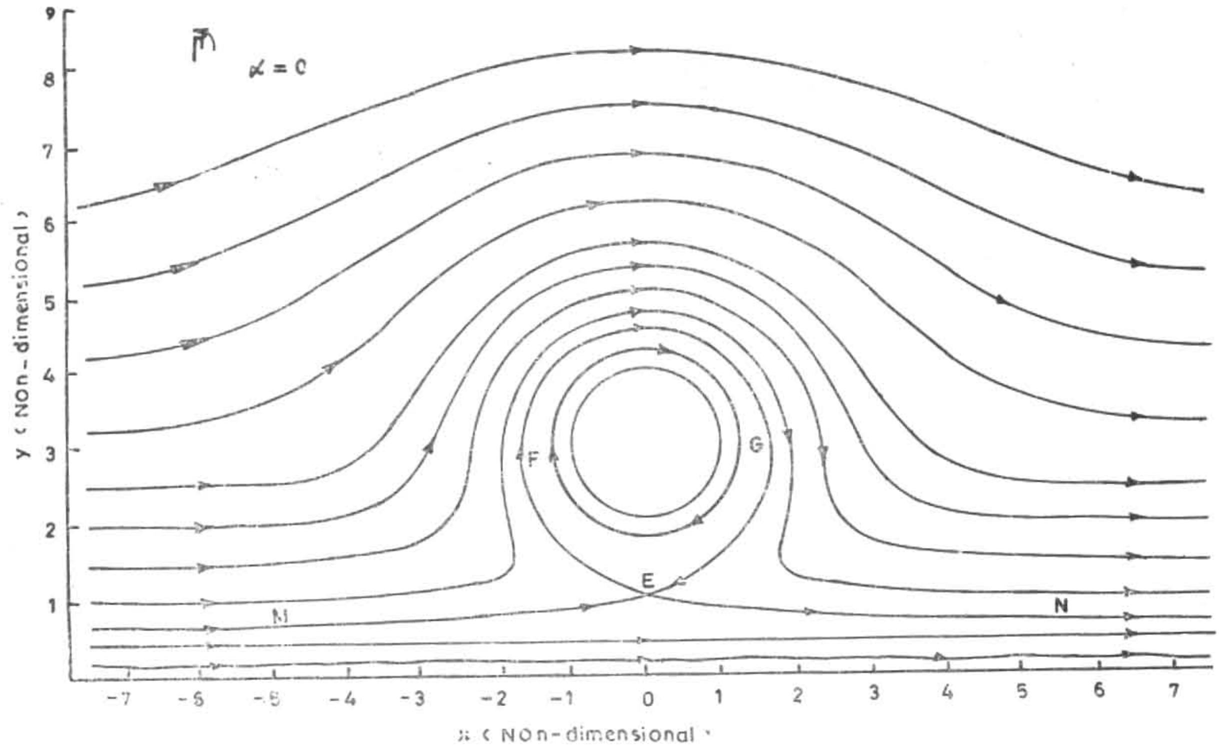

Fig. 3. Path of a positive vortex in absence of any flow

tion and transforming from ζ to z with Eqn. (1), the complex potential W_2, in z-plane, can be written as:

$$W_2 = i\Gamma \ln \sinh\left\{ \frac{\pi}{2\eta_0} \left(\cot^{-1} \frac{z}{-g} - \cot^{-1} \frac{z_0}{-g} \right) \right\} - i\Gamma \ln \sinh\left\{ \frac{\pi}{2\eta_0} \left(\cot^{-1} \frac{z}{-g} - \cot^{-1} \frac{\bar{z}_0}{-g} \right) \right\} \quad (11)$$

where, z_0, is the position of the vortex of strength Γ in z-plane, and $\bar{z}_0$, is complex conjugate of z_0.

Therefore, the complex potential W, for the motion of the fluid due to uniform stream, U, at inifinity and a vortex of strength, Γ at $z=z_0$ is given by adding Eqns. (9) and (11).

$$W = -Uz - 2Ug^2 z \sum_{n=1}^{\infty} \frac{\operatorname{cosech}^2 2n\eta_0}{z^2 + g^2 \coth^2 2n\eta_0} + i\Gamma \ln \sinh\left\{ \frac{\pi}{2\eta_0} \left(\cot^{-1} \frac{z}{-g} - \cot^{-1} \frac{z_0}{-g} \right) \right\} - i\Gamma \ln \sinh\left\{ \frac{\pi}{2\eta_0} \left(\cot^{-1} \frac{z}{-g} - \cot^{-1} \frac{\bar{z}_0}{-g} \right) \right\} \quad (12)$$

4. Equations for path of vortex

The complex potential W_0, for the motion of the vortex itself, can be obtained by subtracting its own potential, $i\Gamma \ln (z - z_0)$ from Eqn. (12), and making $z \to z_0$, using L'Hospital's theorem in third term, omitting additive constant and replacing z_0, by current coordinate z, W_0 can be written in the following form :

$$W_0 = -Uz - 2Ug^2 z \sum_{n=1}^{\infty} \frac{\operatorname{cosech}^2 2n\eta_0}{z^2 + g^2 \coth^2 2n\eta_0} + i\Gamma \ln \frac{\pi}{2\eta_0} \cdot \frac{g^2}{z^2 + g^2} - i\Gamma \ln \sinh\left\{ \frac{\pi}{2\eta_0} \cot^{-1} \frac{z\bar{z} + g^2}{gz - g\bar{z}} \right\} \quad (13)$$

The stream function χ, for the motion of the vortex itself is :

$$\chi = - Uf - \Gamma \ln p_1 p_2 \quad (14)$$

where, $f = y - 2y \sum_{n=1}^{\infty}{}' \, [g^2 \operatorname{cosech}^2 2n\,\eta_0 \, (x^2 + y^2 - g^2 \coth^2 2n\,\eta_0] \,/\, [(x^2 + y^2 - g^2 \coth^2 2n\,\eta_0)^2 + 4g^2 x^2 \coth^2 2n\,\eta_0]$

$$p_1 = \sin \frac{\pi}{4\eta_0} \ln \frac{(y+g)^2 + x^2}{(y-g)^2 + x^2}$$

$$p_2 = \frac{\{(x^2 - y^2 + g^2)^2 + 4x^2 y^2\}^{\frac{1}{2}}}{g^2}$$

Now to reduce Eqn. (14), to non-dimensional form, we will refer the length to the radius of the island, a and time to a suitable time t_0, so that

$$\chi = \frac{a^2}{t_0} \chi' \qquad x = ax'$$

$$U = \frac{a}{t_0} U' \qquad y = ay'$$

$$\Gamma = \frac{a^2}{t_0} \Gamma' \qquad g = ag'$$

With the above relations and since the value of p_1 and p_2 does not alter in non-dimensional form, Eqn. (14) can be written as :

$$-\frac{\chi'}{\Gamma'} = \frac{U'af'}{\Gamma'} + \ln p_1 p_2$$

$$\text{Putting } e^{-\chi'/\Gamma'} = \lambda \text{ and } \frac{U'a}{\Gamma'} = \alpha,$$

$$\lambda = e^{\alpha f'} p_1\, p_2 \tag{15}$$

Eqn. (15) gives the entire family of paths of the vortex in the z-plane.

5. Velocity components for the vortex

The velocity components u and v of the vortex parallel to the plane wall and perpendicular to it respectively, derived from the stream function are given by :

6. Technique for drawing the paths of vortex

As the expression for the stream function for the paths of the vortex, λ, is very cumbersome, it is difficult to draw the paths by pre-assigning constant value of λ. In the expression for λ, there are two parameters, *viz.*, η_0 and α; η_0 depends upon the radius of the island and its distance from the plane boundary and α depends upon the strength and sign of the vortex and uniform stream. Therefore, for a particular size of the island placed at a known distance from the plane boundary, and with a given ratio of the uniform stream and vortex strength, the parameters will have specific values, and so λ, will be function of coordinate points only. Thus, putting values of different coordinate points in Eqn. (15), we can determine the value of λ at a number of points and joining points having same value of λ, we can draw the paths of vortex at any suitable interval of λ, using the method of interpolation.

Taking a = one unit of length, and $b = 3$ units of length, the paths of vortex have been calculated numerically for some typical values of α, *viz.*, $\alpha = 0, \pm 0.5$, as shown in Fig. 3 ($\alpha = 0$), in Fig. 4 ($\alpha = 0.5$) and in Fig. 5 ($\alpha = -0.5$).

7. Discussion of the results

Case 1 : $\alpha = 0$, *i.e.*, $U = 0$ *(Fig. 3)*

The pattern of the paths of vortex will not be altered due to change of the strength of the vortex; only the speed will vary for different strengths. In this case, there exists a stationary point at E, on the imaginary axis between the wall and the island. The vortex makes loops around the island. If it is within the path EFGE, it cannot go out of it, and moves in closed paths. Outside the path EFG, all paths are open and the sense of movement of vortex depends upon its sign. On the imaginary axis, between the wall and island, the vortex moves in opposite directions on opposite side of the stationary point. The paths. ME, EFGE and EN divide the entire region into three distinct sectors. The paths nearer the straight

$$u = -\frac{\partial \chi}{\partial y} = U -$$

$$- 2U \sum_{n=1}^{\infty} \frac{g^2 \operatorname{cosech}^2 2n\eta_0 [(x^2+y^2)^2 (x^2 - y^2) - g^4 (x^2 - y^2) \coth^4 2n\eta_0 - g^6 \coth^6 2n\eta_0] + g^2 (x^4 + y^4 + 10x^2y^2) \coth^2 2n\eta_0}{\{(x^2 + y^2)^2 + 2g^2 (x^2 - y^2) \coth^2 2n\,\eta_0 + g^4 \coth^4 2n\,\eta_0\}^2} +$$

$$+ \frac{\Gamma}{\{(y+g)^2 + x^2\}\{(y-g)^2 + x^2\}} \left[2y\,(x^2 + y^2 - g^2) + \frac{\pi g}{\eta_0} (x^2 + g^2 - y^2) \cot \left\{ \frac{\pi}{4\eta_0} \ln \frac{(y+g)^2 + x^2}{(y-g)^2 + x^2} \right\} \right] \tag{16}$$

$$v = \frac{\partial \chi}{\partial x} = -2U g^2 xy \sum_{n=1}^{\infty} \operatorname{cosech}^2 2n\eta_0 \frac{\{(x^2 + y^2)^2 - 2g^2 \coth^2 2n\,\eta_0 (x^2 - y^2) - 3g^4 \coth^4 2n\,\eta_0\}}{\{(x^2 + y^2)^2 + 2g^4 \coth^2 2n\,\eta_0 (x^2 - y^2) + g^4 \coth^4 2n\,\eta_0\}^2} -$$

$$- \frac{2\Gamma x}{\{(y+g)^2 + x^2\}\{(y-g)^2 + x^2\}} \left[(x^2 + y^2 + g^2) - \frac{\pi g\, y}{\eta_0} \cot \left\{ \frac{\pi}{4\eta_0} \ln \frac{(y+g)^2 + x^2}{(y-g)^2 + x^2} \right\} \right] \tag{17}$$

Since, g is a constant; when $|z| \to \infty$, $u \to U$ and $v \to 0$. On y-axis, v is identically zero.

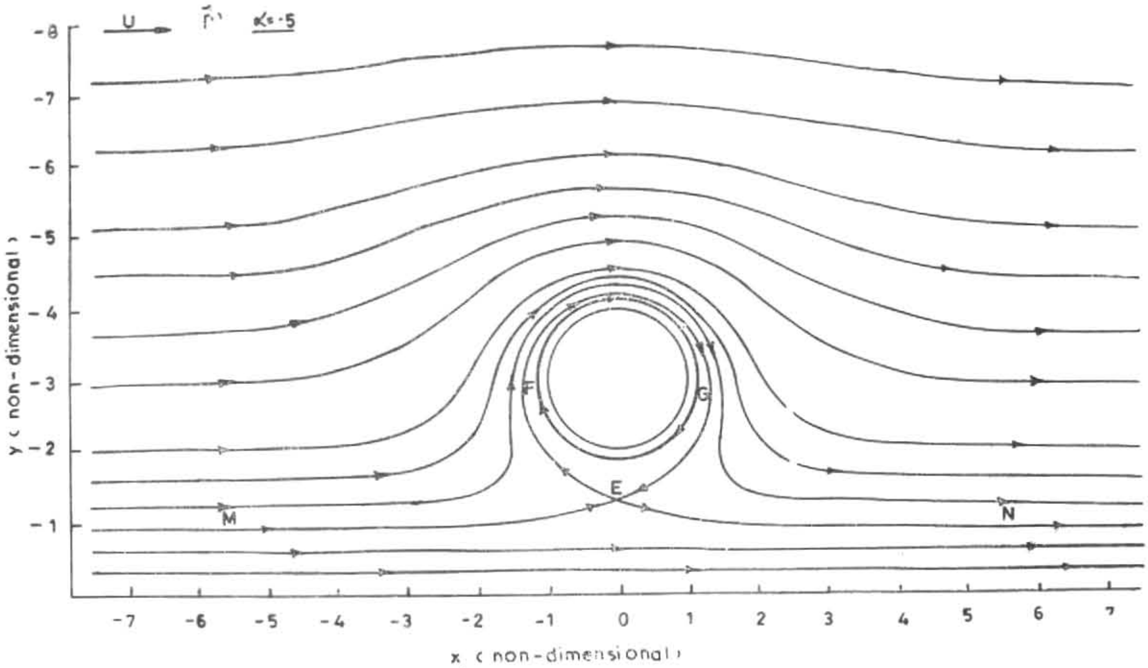

Fig. 4. Path of a positive vortex in uniform stream U, from left to right

boundary or away from the island are comparatively straight.

Case 2 : $\alpha = 0.5$ (*Fig. 4*), α *positive*

When the ratio of the uniform stream and the vortex is positive, the pattern of the paths is similar to those when it is zero, *i.e.*, $\alpha=0$. A stationary point always does exist on the imaginary axis between the wall and island. The position of the stationary point shifts towards the island with increasing value of α, and the region of loop around the island decreases.

Case 3 : $\alpha = -0.5$ (*Fig. 5*), α *negative*

This case is very interesting and significant. Two stationary points exist on the imaginary axis; one between the plane wall and island, and the other away from the island. The parameter λ, has its minimum value zero on the straight wall, on the island and at y, equal to infinity, *i.e.*, at a very large distance from the wall. A critical curve of partial maxima of λ, with dashes, has been drawn through the stationary point F, between the straight wall and the island. The tangential velocity of the vortex along this line is zero. The paths divide the region into six distinct sectors. In the sector KPR, the path of the vortex is closed and makes loops around the island. At stationary point F, the vortex meets from opposite direction along the paths EF and NF and move away from it along the paths FG and FM. The sectors EFM and GFN are symmetrical about the imaginary axis and the vortex makes loops about the critical line in opposite sense extending to infinity on both sides. The vortex in one sector cannot come to another sector. Above and below the critical curve, the sense of movement of the vortex is in opposite directions. Below the path EFG, the paths are open and the vortex moves in the opposite direction of uniform stream. At the second stationary point K, away from the wall and island, the vortex moves towards

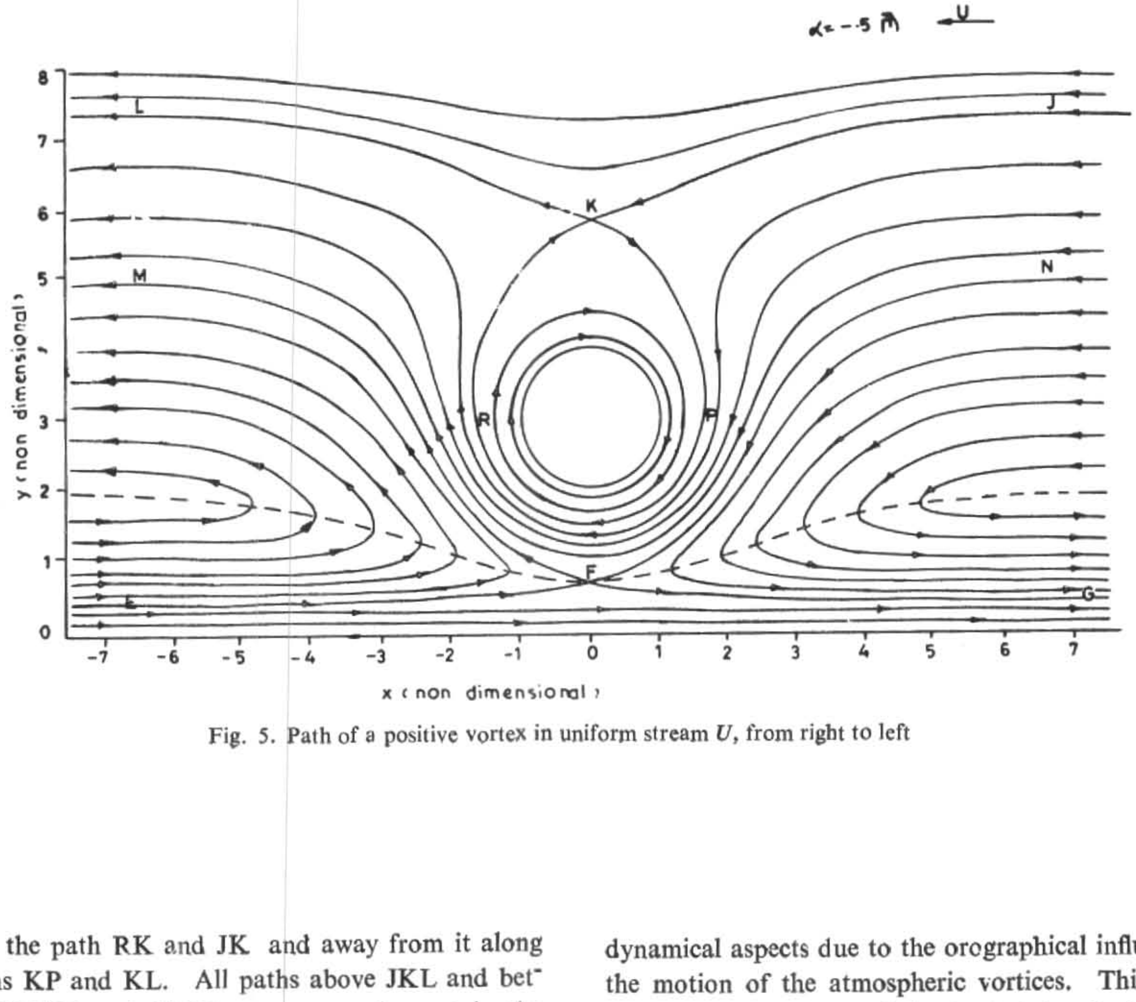

Fig. 5. Path of a positive vortex in uniform stream U, from right to left

it along the path RK and JK and away from it along the paths KP and KL. All paths above JKL and between JKPRKL and NFM are open and move in the same sense as the uniform stream. With the increasing absolute value of α, the loop forming region around the island decreases, the stagnation points between the plane wall and island and away from the island come closer to the wall and island respectively. The critical curve also shifts to the plane wall. Nearer the plane wall or far away from the island the paths are nearly parallel to the plane wall.

8. Remarks

As this problem has been dealt with for perfect fluid in two dimensions, it is a first step in the study of the dynamical aspects due to the orographical influences on the motion of the atmospheric vortices. This investigation may throw some light in studying the behaviour of the motion of the cyclone or low pressure system in the vicinity of mountain ranges.

The approach in this paper can be extended to the study of motion of more than one vortex in the vicinity of multiple boundaries and also for an elliptical island.

Acknowledgements

The author takes the opportunity to express his indebtedness to Dr. K.R. Singh and Dr. B.C. Deka, Professors, Department of Mathematics, Gauhati University for their help and valuable suggestions during the preparation of this paper.

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