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Comparison between two analytical solutions of advection-diffusion equation using separation technique and Hankel transform

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सार – इस कार्य पर एडवेक्शन-डिफ्यूजन समीकरण के दो विश्लेषणात्मक और संख्यात्मक समाधानों के बीच कंट्रास्ट पूरा किया गया है। हम चरों को अलग करने की विधि, हैंकेल ट्रांसफॉर्म और एडोमियन न्यूमेरिकल विधि का उपयोग करते हैं। साथ ही, संयोजन को साफ़ करने के लिए फूरियर पुनर्विक्रय, और वर्ग पूरक विधियों का उपयोग किया गया है। मौजूदा संस्करण अस्थिर परिस्थितियों में इंशास में रेडियोधर्मी आयोडीन-135 (I135) के मिस्र के परमाणु ऊर्जा प्राधिकरण परीक्षण में प्राप्त सूचना सेट के साथ मान्य है। इस मॉडल पर हवा की गति और ऊर्ध्वाधर एड़ी प्रसार को तकनीकों में ऊर्ध्वाधर ऊंचाई की विशेषता के रूप में लिया जाता है और हवा की गति में कार्य के रूप में क्रॉसविंड एडी डिफ्यूसिटी को कार्य के रूप में लिया जाता है। अनुमानित और संख्यात्मक सांद्रता के इन मूल्यों की तुलना ग्राफिक और सांख्यिकीय रूप से देखे गए डेटा से की जा रही है।

ABSTRACT. On this work, contrast between two analytical and numerical solutions of the advection-diffusion equation has been completed. We use the method of separation of variables, Hankel transform and Adomian numerical method. Also, Fourier rework, and square complement methods has been used to clear up the combination. The existing version is validated with the information sets acquired at the Egyptian Atomic Energy Authority test of radioactive Iodine-135 (I^{135}) at Inshas in unstable conditions. On this model the wind speed and vertical eddy diffusivity are taken as characteristic of vertical height in the techniques and crosswind eddy diffusivity as function in wind speed. These values of predicted and numerical concentrations are comparing with the observed data graphically and statistically.

Key words – Advection-Diffusion equation, Separation of Variables and Hankel Transform, Square complement Method.

1. Introduction

An analytical solution of the advection-diffusion equation is obtained using strong assumptions about the eddy diffusivity coefficients and wind speed profiles. They are assumed as constant throughout the whole Atmospheric Boundary Layer (ABL) or follow a power law (van Ulden, 1978; Pasqual and Smith, 1983; Seinfeld, 1986; Tirabassi *et al*., 1986; Sharan *et al*., 1996). Moriera *et al*., (2005) presented a solution of the advection-diffusion equation based on the Laplace transform considering the ABL as a multilayer system.

Essa *et al*. (2011) have given outline of two types of eddy diffusivities by analytically in two-dimensional model. Marrouf *et al*. (2015) presented the changes in advection diffusion equation by influence of eddy diffusivity; Essa *et al*. (2020) evaluated the advection-diffusion equation with variable vertical eddy diffusivity and wind speed using Hankel transform.

The Adomian decomposition method (ADM) has been applied and deterministic problems in many interesting mathematics and physics areas (Adomian, 1994). Adomain gave a review of the decomposition method in (Adomain, 1988). Wazwaz (2001) found the numerical solution of sixth order boundary value problem by ADM, El-Sayed and Abdel-Aziz (2003) compared between Adomian decomposition method and wavelet-Galerkin method for solving integral-differential equations.

In this paper, comparing between two analytical solutions and numerical solution of the advection-diffusion equation has been done using the method of Separation of variables, Hankel Transform, Fourier

transform and square complement and Adomian decompositions method have been used to solve the integration. In this model the wind speed and vertical eddy diffusivity are treated as function of vertical height in the two methods and discretized into *N* sub-interval layers in numerical method. The proposed concentrations are validated with the concentrations data sets obtained from Egyptian Atomic Energy Authority experiment of radioactive Iodine-135 (I^{135}) in unstable conditions.

2. The first mathematical model

The Diffusion equation in three dimensions is

$$u\frac{\partial C(x,y,z)}{\partial x}=\frac{\partial}{\partial y}\left[k_y\frac{\partial C(x,y,z)}{\partial y}\right]+\frac{\partial}{\partial z}\left[k_z\frac{\partial C(x,y,z)}{\partial z}\right] \quad (1)$$

Where, $C(x, y, z)$ is the concentration of pollutants (g/m^3) or (Bq/m^3), k_y and k_z are the eddy diffusivities in crosswind and vertical direction respectively, u is the wind speed (m/s), x is downwind distance (m).

By taking crosswind integration with respect to y from $-\infty$ to ∞, one gets diffusion equation in two dimensions as follows:

$$u\frac{\partial C_y(x,z)}{\partial x}=\frac{\partial}{\partial z}\left[k_z\frac{\partial C_y(x,z)}{\partial z}\right] \quad (2)$$

where, $C_y(x, z)$ is the crosswind integrated concentration of pollutants. Eqn. (2) is solved under the boundary conditions as follows:

(a) The condition of null flux is applied at the mixing height.

$$k_z\frac{\partial C}{\partial z}=0 \text{ at } z=h \quad (2a)$$

(a)' The condition of deposition flux is applied on the ground surface

$$k_z\frac{\partial C}{\partial z}=v_d C(x,z) \text{ at } z=0 \quad (2a)'$$

(b) The mass continuity is used.

$$uC_y(0,z)=Q\delta(z-h) \text{ at } x=0 \quad (2b)$$

where, h is the heightof the atmospheric boundary layer (ABL) (m), "*Q*" is the emission rate (g/s) or (*Bq*) and δ is a Dirac delta function.

(c) The crosswind integrated concentration tends to zero as z tends to ∞.

$$Cy(x,z)\to 0 \quad \text{as } z\to\infty \quad (2c)$$

(d) The crosswind integrated concentration vanishes at the mixing height.

$$Cy(x,z)=0 \quad \text{as } z=h \quad (2c)$$

Assuming the wind speed u and the vertical eddy diffusivity k_z are taken as power law in vertical distance "z" as follows:

$$u=\alpha z^p \quad (3)$$

$$k_z=\gamma z^n \quad (4)$$

$$k_y=\frac{\alpha}{\beta}z^p \quad (4)'$$

where, α, β and γ are constants, n and p depending on stability conditions (Irwin 1979). Then Eqn. (2) can be written as:

$$\gamma z^n\frac{\partial^2 C_y(x,z)}{\partial z^2}+\gamma n z^{n-1}\frac{\partial C_y(x,z)}{\partial z}-\alpha z^p\frac{\partial C_y(x,z)}{\partial x}=0 \quad (5)$$

Multiplying Eqn. (5) by $\frac{z^{2-n}}{\gamma}$, then Eqn. (5) becomes:

$$z^2\frac{\partial^2 C_y(x,z)}{\partial z^2}+nz\frac{\partial C_y(x,z)}{\partial z}-\frac{\alpha}{\gamma}z^{2+p-n}\frac{\partial C_y(x,z)}{\partial x}=0 \quad (6)$$

Changing the independent variable z to s by the substitution $s=z^{\frac{2+p-n}{2}}$ then Eqn. (6) becomes:

$$s^2\frac{\partial^2 C_y}{\partial s^2}+\frac{p+n}{2+p-n}s\frac{\partial C_y}{\partial s}-\frac{\alpha}{\gamma}\left(\frac{2}{2+p-n}\right)^2 s^2\frac{\partial C_y}{\partial x}=0 \quad (7)$$

Eqn. (7) can be further, simplified by the substitution $C_y(x, z) = s^m \psi(x, s)$, where $m = \frac{1-n}{2+p-n'}$, then one gets.

$$\frac{\partial^2 \Psi(x,s)}{\partial s^2} + \frac{1}{s}\frac{\partial \psi(x,s)}{\partial s} - \left(\frac{m}{s}\right)^2 \psi(x,s) - \frac{\alpha}{\gamma}\left(\frac{2}{2+p-n}\right)^2 \frac{\partial \psi(x,s)}{\partial x} = 0 \quad (8)$$

Eqn. (8) can be solved for $\psi(x, s)$, by using Hankeltransform Essa *et al*. (2020), which is defined as follows:

$$\mathcal{H}_m[f(s)] = \tilde{f}(\xi) \equiv \int_0^\infty f(s) J_m(\xi s) s ds$$

J_m is a Bessel function of first one of order "m" and the inverse Hankel transform is defined as

$$\mathcal{H}_m^{-1}\left[\tilde{f}(\xi)\right] = f(s) \equiv \int_0^\infty \tilde{f}(\xi) J_m(\xi s) \xi d\xi$$

where, the Bessel differential operator is defined as follows:

$$\Delta_m f(s) \equiv \frac{d^2 f(s)}{ds^2} + \frac{1}{s}\frac{df(s)}{dz} - \left(\frac{m}{s}\right)^2 f(s)$$

The Hankel transform is given by

$$\mathcal{H}_m[\Delta_m f(s)] \equiv -\xi^2 \tilde{f}(\xi)$$

Applying the Hankel transform on Eqn.(8) and assuming $\psi(x, s) = 0$, $s = \frac{\partial \psi(x,s)}{\partial s} = 0$ as $s \to \infty$

$$\mathcal{H}_m\left[\Delta_m \psi(x,s) = \frac{\alpha}{\gamma}\left(\frac{2}{2+p-n}\right)^2 \frac{\partial \psi(x,s)}{\partial x}\right] \quad (9)$$

One gets:

$$-\xi^2 \tilde{\psi}(x,\xi) = \frac{\alpha}{\gamma}\left(\frac{2}{2+p-n}\right)^2 \frac{\partial \tilde{\psi}(x,\xi)}{\partial x} \quad (10)$$

Eqn. (10) has the solution,

$$\tilde{\psi}(x,\xi) = \tilde{\psi}(x,\xi) \exp\left[-\frac{\gamma}{\alpha}\left(\frac{2+p-n}{2}\right)^2 x\xi^2\right] \quad (11)$$

Using the boundary condition Eqn. (2b) then one can get:

$$\psi(0,s) = \frac{Q}{\alpha s^{\left(\frac{1-n+2p}{2+p-n}\right)}} \delta\left(s^{\frac{2}{2+p-n}} - h\right) \quad (12)$$

Applying Hankel transform to Eqn. (12) we obtain:

$$\mathcal{H}_m[\psi(0,s)] = \tilde{\psi}(0,\xi) \equiv \int_0^\infty \psi(0,s) J_m(\xi s) s ds$$

therefore,

$$\tilde{\psi}(0,\xi) = \frac{Q(2+p-n)h^{\left(\frac{1-n}{2}\right)}}{2\alpha} J_m\left[\xi h^{\left(\frac{2+p-n}{2}\right)}\right]$$

Then Eqn. (11) becomes:

$$\tilde{\psi}(x,\xi) = \frac{Q(2+p-n)h^{\left(\frac{1-n}{2}\right)}}{2\alpha} J_m\left[\xi h^{\left(\frac{2+p-n}{2}\right)}\right] \exp\left[-\frac{\gamma}{\alpha}\left(\frac{2+p-n}{2}\right)^2 x\xi^2\right] \quad (13)$$

Now assuming the inverse of Hankel transformation to Eqn. (13)

$$\mathcal{H}_m^{-1}[\tilde{\psi}(x,\xi)] = \psi(x,s) \equiv \int_0^\infty \tilde{\psi}(x,\xi) J_m(\xi s) \xi d\xi$$

Then, one gets:

$$\psi(x,s) = \frac{Qh^{\left(\frac{1-n}{2}\right)}}{\gamma(2+p-n)x} \exp\left\{-\frac{\alpha\left[h^{(2+p-n)} + s^2\right]}{\gamma(2+p-n)^2 x}\right\} I_m\left[\frac{2\alpha s h^{\left(\frac{2+p-n}{2}\right)}}{\gamma(2+p-n)^2 x}\right] \quad (14)$$

where, I_m is the modified Bessel function of the first kind of order m.

By using the inverse substitution $s = z^{\frac{2+p-n}{2}}$ and $\psi(x, s) = s^{-m} C_y(x, z)$, where $m = \frac{1-n}{2+p-n}$, then the final solution of Eqn. (2) is in the form:

$$C_y(x,z)=\frac{Q(z\ h)^{\frac{1-n}{2}}}{\gamma(2+p-n)x}\exp\left\{-\frac{\alpha\left[z^{(2+p-n)}+h^{(2+p-n)}\right]}{\gamma(2+p-n)^2x}\right\} I_{\left(\frac{1-n}{2+p-n}\right)}\left[\frac{2\alpha(z\ h)^{\left(\frac{2+p-n}{2}\right)}}{\gamma(2+p-n)^2x}\right] \quad (15)$$

and then the concentration in three dimensions has been used as follows:

$$C_y(x,y,z)= \frac{Q(z\ h)^{\frac{1-n}{2}}}{\sqrt{2\pi}\sigma\gamma(2+p-n)x}\exp\left\{-\frac{\alpha\left[z^{(2+p-n)}+h^{(2+p-n)}\right]}{\gamma(2+p-n)^2x}\right\} I_{\left(\frac{1-n}{2+p-n}\right)}\left[\frac{2\alpha(z\ h)^{\left(\frac{2+p-n}{2}\right)}}{\gamma(2+p-n)^2x}\right] e^{-\frac{y^2}{2\sigma_y^2}-\frac{vx}{u}} \quad (16)$$

where, σ_y is the standard deviation in crosswind direction and $e^{-\frac{vx}{u}}$ is the radioactive decay for the specified nuclide (Iodine-135) and v is the decay coefficient of Iodine-135.

3. The second mathematical model

Now the advection-diffusion equation in two dimensions Eqn. (2) will be solved by second mathematical model, considering the height of ABL (h) is discretized into N sub-interval layers such that within each interval, k_z and u are taken as average values. Then the solution of Eqn. (2) is reduced to the solutions of N equations of the following type

$$u_i\frac{\partial C_y(x,z)}{\partial x}=k_i\frac{\partial^2 C_y(x,z)}{\partial z^2}$$

where,

$$k_i\frac{1}{z_{i+1}-z_i}\int_{z_n}^{z_{i+1}}k_i(z)dz$$

$$u_i\frac{1}{z_{i+1}-z_i}\int_{z_i}^{z_{i+1}}u_i(z)dz$$

for, $z_i \le z \le z_{i+1}$, $i = 1 : N$

By using separation of variables, assuming the general solution of Eqn. (17) in the form:

$$Cy\ (x, z, h) = X\ (x)\ Z\ (z, h) \quad (18)$$

Substituting from Eqn. (18) in Eqn. (17) and dividing on $X\ (x)\ Z\ (z, h)$, one can get:

$$\frac{1}{X(x)}\frac{dX(x)}{dx}=\frac{k_i}{u_iZ(z,h_s)}\frac{d^2Z(z,h)}{dz^2}=-\lambda^2 \quad (19)$$

where, λ^2 is a constant of separation. Eqn. (19) is divided into the following two equations as follows:

$$\frac{dX(x)}{dx}=-\lambda^2X(x) \quad (19a)$$

$$\frac{d^2Z(z,h)}{dz^2}=-\frac{u_n\lambda^2}{k_n}Z(z,h) \quad (19b)$$

The solutions of Eqns. (19a, 19b) have the form:

$$X(x)=c(h)e^{-\lambda_l^2x} \quad (20)$$

$$Z(z,h)=A_1(h)e^{i\lambda z\sqrt{\frac{u_i}{k_i}}}+A_2(h)e^{-i\lambda z\sqrt{\frac{u_i}{k_i}}} \quad (21)$$

where, $c\ (h)$, $A_1\ (h)$ and $A_2\ (h)$ are depending on mixing height (h). Then the solution of Eqn. (18) can be written as follows:

$$C_{yn}(x,z,h)=c(h)A_1(h)e^{-\lambda_l^2x+i\lambda_lz\sqrt{\frac{u_i}{k_i}}}+c(h)A_2(h)e^{-\lambda_l^2x+i\lambda_lz\sqrt{\frac{u_i}{k_i}}} \quad (22)$$

Since $0 < \lambda_l < \infty$, where, l = 0, 1, 2, …., varies continuously as integer values, the sum of all these solutions depends on the integration of λ_l so the general solution is as follows:

$$C_{yn}(x,z,h)=\int_0^\infty\left[c(\lambda_1,h)A_1(\lambda_l,h)e^{-\lambda_l^2x+i\lambda_lz\sqrt{\frac{u_i}{k_i}}}+c(\lambda_1,h)A_2(\lambda_1,h)e^{-\lambda_l^2x+i\lambda_lz\sqrt{\frac{u_i}{k_i}}}\right]d\lambda_i \quad (23)$$

Also, we can write Eqn. (23) in the form

$$C_{yn}(x,z,h)=\int_0^{\infty}\Big[c(\lambda_1,h)A_1(\lambda_l,h) + c(-\lambda_1,h)A_2(-\lambda_l,h)e^{-\lambda_l^2 x+i\lambda_l z\sqrt{\frac{u_i}{k_i}}}d\lambda_l\Big] \quad (24)$$

Let,

$$R(\lambda_l,h)=\left[c(\lambda_1,h)A_1(\lambda_l,h)+c(-\lambda_1,h)A_2(-\lambda_l,h)\right]$$

such that

$$R(\lambda_l,h)=\left[c(\lambda_1,h)A_1(\lambda_l,h)\right] \quad if \ \ \lambda_l>0$$

$$R(\lambda_l,h)=\left[c(-\lambda_1,h)A_2(-\lambda_l,h)\right] \quad if \ \ \lambda_l<0$$

then, Eqn. (24) becomes

$$C_{yn}(x,z,h)=\int_{-\infty}^{\infty} R(\lambda_1,h)e^{-\lambda_l^2 x+i\lambda_l z\sqrt{\frac{u_i}{k_i}}}d\lambda_l \quad (25)$$

To find the value of R (λ_l, h) use the Fourier Transform of δ ($z-h$) as follows:

Then, Fourier Transform of δ ($z-h$) is

$$\delta(z-h)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda_l(z-h)\sqrt{\frac{u_i}{k_i}}}d\lambda_l \quad (26)$$

By using the boundary condition in Eqn. (2b) then, the value of R (λ_l, h) can be written as follows:

$$R(\lambda_l,h)=\frac{Q}{2\pi u_i}e^{i\lambda_l h_s\sqrt{\frac{u_i}{k_i}}} \quad (27)$$

Then Eqn. (25) can be written as follows:

$$C_{yn}(x,z,h)=\frac{Q}{2\pi u_n}\int_{-\infty}^{\infty} e^{i\lambda_l^2 x+i\lambda_l(z-h)\sqrt{\frac{u_i}{k_i}}}d\lambda_l \quad (28)$$

Considering the square compliment method to solve the above integration Essa *et al*. (2011), then the solution of Eqn. (17) can be written as follows:

$$C_{yn}(x,z,h)=\frac{Q}{2u_i\sqrt{\pi x}}e^{-\frac{(z-h)^2 u_i}{4k_i x}} \quad (29)$$

then the concentration in three dimensions will be

$$C(x,y,z,h)=\frac{Q}{2\sqrt{2x}\pi\sigma_y u_i}e^{-\frac{(z-h)^2 u_i}{4k_i x}-\frac{y^2}{2\sigma_y^2}-\frac{\nu x}{u}} \quad (30)$$

where, u_i and k_i are taken from two equations (3) and (4) respectively. $\sigma_y$is the standard deviation in y direction and $e^{\frac{\nu x}{u}}$ is the radioactive decay for the specified nuclide, ν is radioactive coefficient.

Numerical method

Now the advection-diffusion equation in three dimensions Eqn. (2) will be solved by second mathematical model, considering the height of ABL (h) is discretized into *N* sub-interval layers such that within each interval, k_z, k_y and u are taken as average values. Then the solution of Eqn. (2) is reduced to the solutions of *N* equations of the following type:

$$u_n\frac{\partial C(x,z)}{\partial x}=k_n\frac{\partial^2 C(x,z)}{\partial z^2}+k_n\frac{\partial^2 C(x,z)}{\partial y^2} \quad (31)$$

where,

$$k_n=\frac{1}{z_{n+1}-z_n}\int_{z_n}^{z_{n+1}} k_n(z)dz$$

$$u_n=\frac{1}{z_{n+1}-z_n}\int_{z_n}^{z_{n+1}} u_n(z)dz$$

for,$z_n\le z\le z_{n+1}$, i = 1 : N

$$\frac{\partial^2 C(x,z)}{\partial z^2}=\frac{u_n}{k_{zn}}\frac{\partial C(x,z)}{\partial x}-\frac{k_{yn}}{k_{zn}}\frac{\partial^2 C(x,z)}{\partial y^2} \quad (32)$$

Taking $A = u_n/ k_{zn}$ and $B = k_{yn}/k_{zn}$ and $k_{yn} = \beta u_n$

Equation (32) can be solved using Adomian decompositions method as follows:

$$L_{zz}C\,(x,\,z) = A\,L_x\,C\,(x,\,z) - BL_{yy}C\,(x,\,z)$$

where, $L_{zz} = \frac{\partial^2}{\partial z^2}$, $L_x = \frac{\partial}{\partial x}$ and $L_{yy} = \frac{\partial^2}{\partial y^2}$

Multiplying both sides of this equation by inverse L_{zz}^{-1}

$$C(x,z) = C_o + A\,L_{zz}^{-1}L_xC(x,z) - BL_{zz}^{-1}L_{yy}C(x,z) \quad (33)$$

where, $L_{zz}^{-1} = \int\int_o^z .....dzdz$

Assuming that :

$$C_o = M\,(x) + zN\,(x) \quad (34)$$

where, M and N are unknown function which will be determined from boundary condition using equation (34) to get the general solution in the from:

$$C_{n+1} = A\int\int_o^z \frac{\partial C_n}{\partial x}dzdz - B\int\int_o^z \frac{\partial^2 C_n}{\partial y^2} \quad (35)$$

Put $n = 0$

$$C_1 = A\int\int_o^z \frac{\partial C_o}{\partial x}dzdz - B\int\int_o^z \frac{\partial^2 C_o}{\partial y^2}$$

$$C_1 = A\int\int_o^z \left(\frac{\partial M}{\partial x} + z\frac{\partial N}{\partial x}\right)dzdz - B\int\int_o^z \left(\frac{\partial^2 M}{\partial y^2} + z\frac{\partial^2 N}{\partial y^2}\right)dzdz$$

$$C_1 = \left(A\frac{\partial M}{\partial x} - B\frac{\partial^2 M}{\partial y^2}\right)\frac{z^2}{2!} + \left(A\frac{\partial N}{\partial x} - B\frac{\partial^2 N}{\partial y^2}\right)\frac{z^3}{3!} \quad (36)$$

Assuming the solution has the form:

$$W_n = \sum_0^\infty C_n$$

$$W_1 = C_0 + C_1 = M + zN + \left(A\frac{\partial M}{\partial x} - B\frac{\partial^2 M}{\partial y^2}\right)\frac{z^2}{2!} + \left(A\frac{\partial N}{\partial x} - B\frac{\partial^2 N}{\partial y^2}\right)\frac{z^3}{3!} \quad (37)$$

By differentiating the equation (37) with respect to z and multiplying by k_z we obtain:

$$k_z\frac{\partial w_1}{\partial z} = k_zN(x) + zk_z\left(A\frac{\partial M}{\partial x} - B\frac{\partial^2 M}{\partial y^2}\right) + \frac{z^2}{2!}k_z\left(A\frac{\partial N}{\partial x} - B\frac{\partial^2 N}{\partial y^2}\right) \quad (38)$$

Using the boundary condition (8c) at $z = 0$, we obtain

$$k_z\frac{\partial w_1}{\partial z} = k_zN(x) = v_dM(x)$$

$$M(x) = \frac{k_o}{v_d}N(x) \quad (39)$$

Using the boundary condition (8b) at $z = h$, we obtain that:

$$N(x,y) + h\left(A\frac{\partial M}{\partial x} - B\frac{\partial^2 M}{\partial y^2}\right) + \frac{h^2}{2}\left(A\frac{\partial N}{\partial x} - B\frac{\partial^2 N}{\partial y^2}\right) = 0 \quad (40)$$

Eqn. (40) becomes:

$$\frac{\left(-h^2Bv_d - 2k_oBh\right)}{2k_o}\left(\frac{\partial^2 M}{\partial y^2}\right) + \left(\frac{h^2Av_d}{2k_o} + hA\right)\left(\frac{\partial M}{\partial x}\right) + \frac{v_d}{k_o}M(x,y) = 0 \quad (41)$$

The final form of Eqn. (41) in the form:

$$\frac{\partial^2 M}{\partial y^2} - \frac{A}{B}\frac{\partial M}{\partial x} - DM = 0 \quad (42)$$

where, $D = \frac{2v_d}{Bh(2k_o + hv_d)}$

TABLE 1

Power-law exponent *p* and *n* of wind speed and eddy diffusivity as a function of air stability in urban area

	A	B	C	D	E	F
p	0.15	0.15	0.20	0.25	0.40	0.60
n	0.85	0.85	0.80	0.75	0.60	0.40

TABLE 2

Shows that the values of standard deviation in crosswind σ_y through different stabilities

Stability classes	Values of σ_y
A	$\sigma_y = 0.40x^{0.91}$
B	$\sigma_y = 0.40x^{0.91}$
C	$\sigma_y = 0.36x^{0.86}$
D	$\sigma_y = 0.32x^{0.78}$

Eqn. (42) is solved by separation method as follows:

$$\frac{dX}{dx} - \frac{\left(D-\lambda^2\right)B}{A}X = 0; \qquad A \neq 0 \tag{43a}$$

$$\frac{\partial^2 Y}{\partial y^2} + \lambda^2 Y = 0 \tag{43b}$$

Then, the solution of Eqn. (43b) is in the form:

$$Y(y) = c_3\cos(\lambda y) + c_4 \sin(\lambda y) \tag{44}$$

Taking the condition $\frac{\partial C}{\partial y} = 0$ at $y = 0$, L_y where, L_y is a large distance in y direction. Then, Eqn. (44) becomes:

$$Y(y) = c_3 \cos\left(\frac{i\pi}{L_y}\right)y$$

Also, the solution of Eqn. (43a) becomes:

$$X(x) = c_5 e^{\frac{\left(D-\lambda^2\right)Bx}{A}}$$

Then, the total solution of Eqn. (42) becomes:

$$M(x,y) = c_6 e^{\frac{\left(D-\lambda^2\right)Bx}{A}}\cos\left(\frac{i\pi}{L_y}\right)y$$

By applying the condition

$uC(x,y,z) = Q\delta(z)\delta(y)$ at $x = 0$, One gets:

$$M(x,y) = \frac{Q}{u}e^{\frac{\left(D-\lambda^2\right)Bx}{A}}\cos\left(\frac{i\pi}{L_y}\right)y \tag{45}$$

Substituting equations (39) and (45) in equation (34), one obtains:

$$C_o = \left(1+\frac{v_d}{k_o}\right)\frac{Q}{u}e^{\frac{\left(D-\lambda^2\right)Bx}{A}}\cos\left(\frac{i\pi}{L_y}\right)y \tag{46}$$

Also, equation (46), becomes:

$$C_1 = (BD)\left(\frac{z^3}{3!}+\frac{v_d}{k_o}\frac{z^2}{2!}\right)M(x,y) \tag{47}$$

where, $D = \frac{2v_d}{Bh(2k_o + hv_d)}$

Similarity, we get

$$C_2 = (BD)^2\left(\frac{z^5}{5!}+\frac{v_d}{k_o}\frac{z^4}{4!}\right)M(x,y)$$

$$C_3 = (BD)^3\left(\frac{z^7}{7!}+\frac{v_d}{k_o}\frac{z^6}{6!}\right)M(x,y) \tag{48}$$

$$C_4 = (BD)^4\left(\frac{z^9}{9!}+\frac{v_d}{k_o}\frac{z^8}{8!}\right)M(x,y)$$

The general solution:

$$C_y(x,y,z) = \frac{Q}{u}\left[\left(1+\frac{v_d}{k_o}\right)+\sum_{i=1}^{n}\left[\frac{z^{2i}}{(2i)!}+\frac{v_d}{k_o}\frac{z^{2i+1}}{(2i+1)!}\right]e^{\frac{\left[D-(i\pi/L_y)^2\right]Bx}{A}+\frac{vx}{u}}\cos\left(\frac{i\pi}{L_y}\right)y\right] \tag{49}$$

where, v is the decay factor of isotope I^{135} which equals $2.9*10^{-5}s^{-1}$.

TABLE 3

Meteorological data of the nine convective test runs at Inshas site in March and May 2006

Run no.	Working hours of the source	Release rate (Bq)	Wind speed (ms^{-1})	Wind Direction(deg)	W_* (ms^{-1})	P-G stability class	H (m)	Vertical distance (m)
1	48	1028571	4	301.1	2.27	A	600.85	5
2	49	1050000	4	278.7	3.05	A	801.13	10
3	1.5	42857.14	6	190.2	1.61	B	973	5
4	22	471428.6	4	197.9	1.23	C	888	5
5	23	492857.1	4	181.5	0.958	A	921	2
6	24	514285.7	4	347.3	1.3	D	443	8.0
7	28	1007143	4	330.8	1.51	C	1271	7.5
8	48.7	1043571	4	187.6	1.64	C	1842	7.5
9	48.25	1033929	4	141.7	2.1	A	1642	5.0

TABLE 4

Observed, calculated and numerical concentrations for Run 9 experiments

Test	Downwind distance (m)	Observed conc.(Bq/m^3)	Predicted conc. One Eqn.(16) (Bq/m^3)	Predicted conc. Two Eqn.(30) (Bq/m^3)	Numerical conc. Three Eqn.(49) (Bq/m^3)
1	100	0.025	0.030	0.010	0.019697
2	98	0.037	0.045	0.011	0.012259
3	136	0.091	0.096	0.045	0.082274
4	135	0.197	0.218	0.163	0.083528
5	106	0.272	0.293	0.196	0.056512
6	186	0.188	0.206	0.128	0.109531
7	165	0.447	0.460	0.322	0.159115
8	154	0.123	0.139	0.094	0.164853
9	106	0.032	0.040	0.016	0.0198

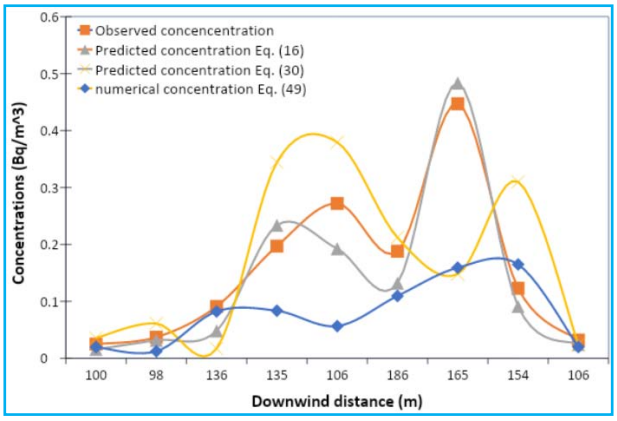

Fig. 1. The variation of concentration (Bq/m^3) for Iodine 135 with downwind distance

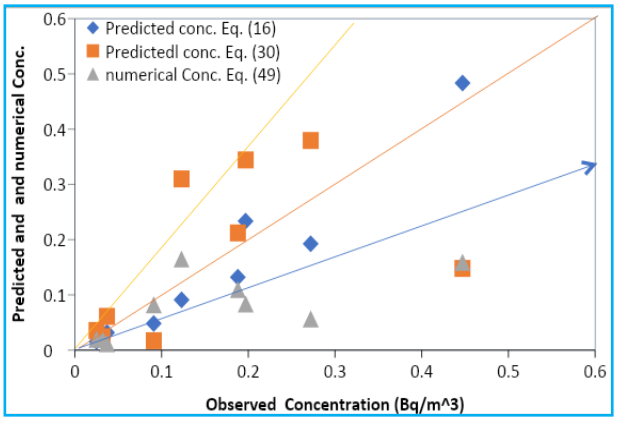

Fig. 2. Scattering diagram between predicted, Numerical and observed concentrations (Bq/m^3) for Iodine-135

TABLE 5

Shows that statistical evaluation of present model in unstable condition

Model	NMSE	FB	COR	FAC2
Predicted model 1	0.008	-0.08	1	1.08
Predicted model 2	0.20	0.36	0.99	0.70
Numerical concentration	1.36	0.67	0.66	0.50

4. Results and discussion

The observed data of I^{135} isotope concentration was obtained from dispersion experiments conducted in unstable condition air samples which was collecting around the Egyptian Atomic Energy authority, Research Reactorat Inshas, Cairo, Egypt. The samples were collected at a height of 0.7m above ground from a stack of height 43 m. The Reactor site was flat and dominated by sandy soil with a poor vegetation cover with a roughness length of 0.6 cm and each run is made through 30 minutes. The values of power-law exponent '*p*' and "*n*" of eddy diffusivity as a function of air stability are taken from Hanna *et al*. (1982) and presented in Table 1. Standard deviation of crosswind σ_y is taken from Hanna *et al*. (1982) and presented in Table 2. The meteorological data of I^{135} isotope during the experiments are taken from Essa and Maha (2007) and presented in Table 3. Eqns. (16) and (30) are estimated using two Eqns. 3 & 4 below the plume center line to compare between two predicted concentrations which are calculated using Mathematica program, Adomian numerical method from Eqn. (49) and observed concentrations date of I^{135} from Research Reactorat Inshas, Cairo, Egypt as in Table 4 as follows:

A comparison between two predicted, numerical and observed concentrations of radioactive I^{135} in unstable condition at Inshas are shown in two Figs. 1 and 2. From these two figures, one finds that the two predicted concentrations lie inside a factor of two with observed concentrations data but most numerical concentration data lie inside a factor of two with the observed concentration data.

5. Model evaluation statistics

The statistical method is presented and comparison between predicted and observed results as offered by Hanna (1989) is done. The following standard statistical performance measures and characterizes the agreement between predictions ($C_p = C_{pred}$) and observations ($C_o = C_{obs}$):

$$\text{Fraction Bias (FB)} = \frac{\left(\overline{C_o} - \overline{C_p}\right)}{\left[0.5\left(\overline{C_o} - \overline{C_p}\right)\right]}$$

Normalized Mean Square Error

$$\text{(NMSE)} = \frac{\overline{\left(C_p - C_o\right)^2}}{\left(\overline{C_p C_o}\right)}$$

Correlation Coefficient (COR)

$$= \frac{1}{N_m}\sum_{i=1}^{N_m}\left(C_{pi} - \overline{C_p}\right) \times \frac{\left(C_{oi} - \overline{C_o}\right)}{\left(\sigma_p \sigma_o\right)}$$

$$\text{Factor of Two (FAC2)} = 0.5 \le \frac{C_p}{C_o} \le 2.0$$

where, σ_p and σ_o are the standard deviations of predicted C_p and observed C_o concentrations, respectively. Over bars refer to the average over all measurements. A perfect model must have the following performance: NMSE = FB = 0 and COR= FAC2 = 1.0.

One can easily see from Table 5, the statistical technique shows that the proposed model Predicted one is very well agreement with observed data concentrations than predicted model two, also, the numerical concentration is less agreement with observed concentration according to NMSE and FB are near to zero, COR and FAC2 are close to one. The predicted model one is well agreement with observed model than predicted model two and numerical concentration model.

6. Conclusions

We have an analytical solution of three-dimensional atmospheric diffusion equation by the method of Separation of variables, Hankel transform and Adomian numerical method to calculate concentration for Iodine-135. In this model the wind speed and vertical eddy diffusivity are treated as function of vertical height and the crosswind eddy diffusivity as function in wind speed. The predicted model one is one to one with observed concentrations data than predicted model two and numerical model. Two predicted models are inside a factor of two with the observed concentration than numerical model. Also regarding to NMSE and FB are near to zero, also, COR and FAC2 are close to one. The predicted one is well agreement with observed concentration than predicted two and numerical model.

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