

Determination of fractal dimension of chaotic attractor for maximum temperature over Madras

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सार — निर्धारणालक केओस विधि द्वारा क्रमावस्था अंतरिक्ष प्रक्षेप-पथों से मद्रास की दो वेधशालाओं, नामतः नुंगम्बक्कम और मीनाम्बक्कम, में रिकार्ड किए गए अधिकतम दैनिक तापमान (मार्च-मई के दौरान) आकर्षियों (अट्रैक्टर्स) के आयामों का इसमें आकलन किया गया है। ये आयाम सम्मिश्र गतिकीय पद्धति को समझने के लिए अपेक्षित न्यूनतम प्राचलों के लिए मूलभूत सूचना देते हैं और इस पद्धति का मॉडल तैयार करने के लिए अपेक्षित प्राचलों की अधिकतम पर्याप्त सीमा का भी इनसे पता चलता है। मौसम परिघटना के आंशिक आयाम नामतः मद्रास का अधिकतम तापमान 3.5 से 3.9 के बीच रहने से इस बात का पता चलता है कि इस पद्धति का मॉडल तैयार करने के लिए कम से कम 4 प्राचलों की आवश्यकता है और अधिकतम 19 प्राचल पर्याप्त हैं।

ABSTRACT. The dimensions of attractors of daily maximum temperature (during March-May) recorded by the two observatories of Madras, viz, Nungambakkam and Meenambakkam are estimated from phase space trajectories by the method of deterministic chaos. The dimensions provide the basic information on the minimum number of parameters required to understand the complex dynamical system and also the upperbound (degrees of freedom) of such parameters that are sufficient to model the system. The fractal dimension for the weather event, viz, maximum temperature over Madras is between 3.5 and 3.9 suggesting 4 parameters are necessary to model the system and a maximum of 19 parameters are sufficient.

Key words — Deterministic chaos, Fractal dimension, Embedding space, Dimensionality, Phase space, Strange attractors.

1. Introduction

The minimum number of independent variables, required to understand the dynamics underlying complex systems, is being obtained by the theory of deterministic chaos, a science which has been developed during the last two decades. Such studies were made for Indian monsoon rainfall (Satyan 1988, Sujit Basu and Andharia 1992) and earthquake sequence (Battacharya *et al.* 1995).

A system whose status changes with time is known as dynamical system. The dissipative dynamical system (which loses energy due to friction) always approaches a limiting or asymptotic state over time which is the main focus of the theory of deterministic chaos. The concepts of the theory of chaos are excellently described elsewhere (Kaye 1993, Gleick 1987 and Chatterjee and Yilmaz 1992). A complex dynamical system whose instantaneous status at different points of time is represented in phase space ultimately converge to a set called an attractor. The dimension of the attractor (need not be an integer) is far less than that

of the phase space. The fractals, a concept from pure geometry, exhibit self-similarity at levels of transformation is another attracting area of chaos and they may have non-integer dimension which is called the fractal dimension of chaotic attractor (Mandelbrot 1977).

Forecasting maximum temperature over Madras poses problem to the forecaster in view of the modifying effects like time of onset, intensity and the extent of sea breeze circulation, strength of continental airmass advection etc. The sea breeze effect over Madras is more pronounced and clearly identified during the pre-monsoon season (March-May) and south-west monsoon season (June-September) due to reversal of zonal flow. However, the same cannot be easily identified as such during the other seasons as there is considerable amount of complexity and uncertainty in identifying the onset and maintenance of sea breeze circulation since the zonal flow being unilateral during these periods (Atkinson 1981). In this study we propose to find out the fractal dimension of chaotic attractor for daily maximum temperature over Madras for the pre-monsoon season.



Fig.1. Geographical locations of meteorological observations of Chennai

2. Theoretical background

A simple deterministic dynamical system of n variables, $x_1, x_2, x_3, \dots, x_n$ is characterized by a set of n differential equations

$$\dot{x}_i = f_i(x_1, x_2, x_3, \dots, x_n) \quad i=1,2,3,\dots,n \quad (1)$$

(the superscript \cdot denotes the time derivative).

The instantaneous description of the system is represented by the vector points $[x_1(t), x_2(t), x_3(t), \dots, x_n(t)]$ at any given time t . The simple time evolution model may look as follows:

$$\begin{aligned} \dot{x} &= f_\omega(x(t)) \text{ in continuous time} \\ x_{t+\tau} &= f_\omega(x_t) \text{ in discrete time} \end{aligned} \quad (2)$$

where ω is some fixed parameter such as the driving force of the system and τ is the time shift.

Let us consider x_t as a vector $[x_1(t), \dots, x_n(t)]$ in Euclidian space R^n and a function $f_\omega: R^n \rightarrow R^n$. If x is observed at some initial time, say, $t = 0$, then

$$x_t = f_\omega(f_\omega(\dots f_\omega(x_0)\dots))f_\omega^t(x_0) \quad (3)$$

Let $U = \{\text{possible states of the system having positive volume (Lebesgue measure)}\}$.

The volume will be compressed due to loss of energy and the set U converges asymptotically to a compact set A if the system is dissipative. A is called an attracting set (also called attractor) if $f_\omega^t(A) = A$ for any t and for every open set $V \supset A$, we have $f_\omega^t(U) \subset V$. The union of inverse images $(f_\omega^t)^{-1}(U)$ for all t is called the basin of attractor of A . The attractors may be a single stable point or periodic with fixed period (e.g. circle) or quasi periodic (e.g., torus) or aperiodic

(chaotic). The chaotic attractors can not be obtained by bounded deformations and diffeomorphisms (invertible and continuously differentiable transformation) and does not have any regular shape and that is why they are called strange attractors (Chatterjee and Yilmaz 1992).

Eqn. (1) can be transformed into a single highly non-linear differential equation

$$x^n = f(x, x', x'', x''', \dots, x^{(n-1)}) \quad (4)$$

where, x is one of the variables of $x_1, x_2, x_3, \dots, x_n$ and all the other variables are eliminated by differentiation. The vector $X(t) = [x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)]$ is treated as single observation in a dimensional phase space. Since the meteorological dynamical system variables are observables (discrete), the vector $X(t) = [x(t), x(t+\tau), x(t+2\tau), \dots, x(t+(m-1)\tau)]$ (in the case of discrete time series) is treated as single observation in an m dimensional phase space and is chosen in such a way that the data points of the vector $X(t)$ and $X(t+\tau)$ are linearly independent. Then the number of pairs of vector points whose Euclidian distance (π_{ij}) is less than a prescribed threshold value (l) are found out by,

$$N(l) = \sum_{i=1}^{N-(m-1)} \sum_{j=i+1}^{N-(m-1)} \theta(l - \pi_{ij}) \quad (5)$$

where $\theta(a) = 0$ if $a < 0$ and $\theta(a) = 1$ if $a > 0$ is the Heavyside function. As there will be a maximum of $[N-(m-1)]$ vector points possible with N data points in m dimensional phase space, the number of distinct pairs of points will be $[N-(m-1)] C_2$. The value $N(l)$ is normalised by dividing it by $[N-(m-1)] C_2$. Then

$$C(l) = N(l) / ([N-(m-1)] C_2) \quad (6)$$

is the cumulative distributive function of the correlation integral. Then $C(l) = a \cdot l^D$ (Grassberger and Procaccia 1983) where a is a constant and the correlation fractal dimension of the attractor is given by,

$$D = (\log [C(l) / \log(l)]) \quad (7)$$

3. Data

Daily maximum temperature of Meenambakkam Airport Meteorological Office (herein after called MO) and Nungambakkam observatory (herein after called ACWC, Area Cyclone Warning Centre) for the period March-May of 1971 to 1985 have been used in this study. The geographical locations of above mentioned observatories are shown in Fig.1.

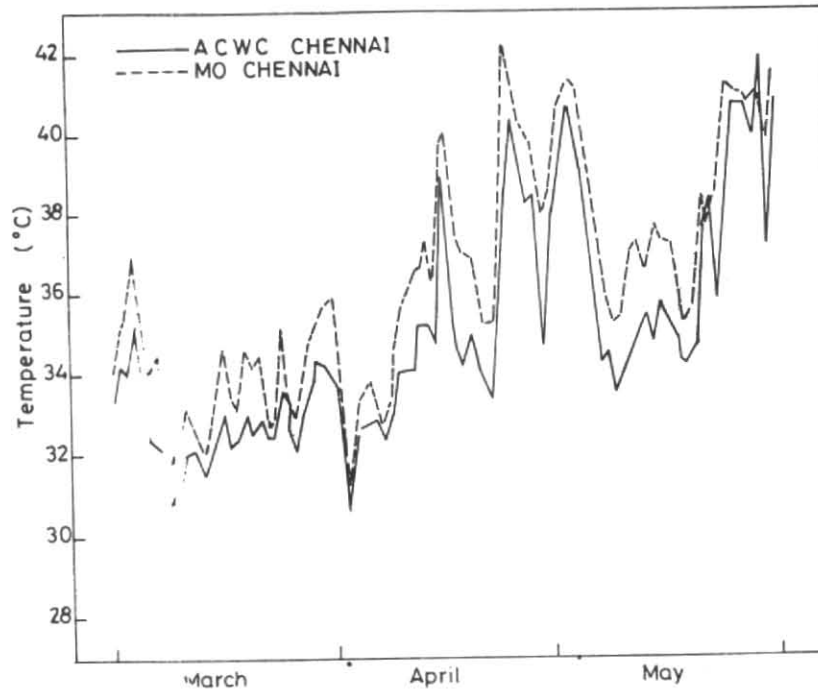


Fig.2. Time series plots of daily maximum temperatures of MO and ACWC Chennai (1 March - 31 May 1985)
Time series plots of daily maximum temperatures of MO and ACWC Chennai (1 March - 31 May 1985)

TABLE 1
Results of spectrum analysis of daily maximum temperature of MO and ACWC Madras during
1 March to 31 May (1971- 1985)

Year	ACWC Madras		MO Madras	
	No. of lags significant	Periodicity (days)	No. of lags significant	Periodicity (days)
1971	1 to 6	—	1 to 6	—
1972	1 to 4	—	1 to 4	—
1973	1 to 5	9.14	1 to 6	—
1974	1 to 5	5.82 & 5.33	1 to 6	5.82 & 5.33
1975	1 to 8	3.76	1 to 8	3.20
1976	1 to 5	21.33 & 16.0, 3.76, 3.56, 3.37, 2.13 & 2.06	1 to 5	21.33 & 16.0
1977	1 to 8	2.29	1 to 7	8.00 & 7.11
1978	1 to 7	2.46, 2.37 & 2.29	1 to 6	2.46, 2.37, 2.29 & 2.21
1979	1 to 3	5.82	1 to 4	—
1980	1 to 5	—	1 to 5	—
1981	1 to 6	2.21, 2.13 & 2.03	1 to 6	2.78
1982	1 to 3	10.67, 9.14 & 8.00	1 to 3	10.67, 9.14 & 8.00
1983	1 to 4	2.21, 2.13 & 2.03	1 to 3	2.21
1984	1 to 5	—	1 to 6	—
1985	1 to 5	2.46 & 2.37	1 to 5	—

4. Methodology and computation

4.1. Time series analysis

The plot of the march of the daily maximum temperature recorded at MO and ACWC Madras during the hot-

weather season (1 March to 31 May) over a typical year is shown in Fig.2. It can be seen that the maximum temperature observed over MO is at least 1°C higher than that observed over ACWC indicating the slight variability of weather parameter over meso v scale (2 to 20 km in horizontal scale)

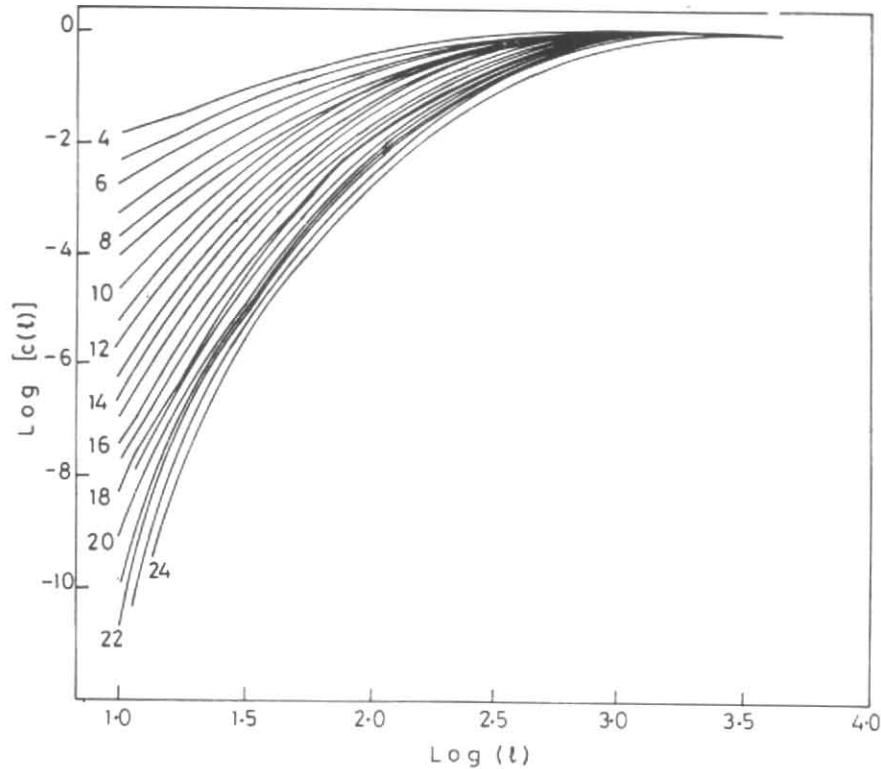


Fig.3. Cumulative distribution function of distances of pre-monsoon season daily maximum temperature trajectory in m -dimensional phase space ($m=4$ to 24) in respect of MO Chennai (1971-85) for $\tau=1$

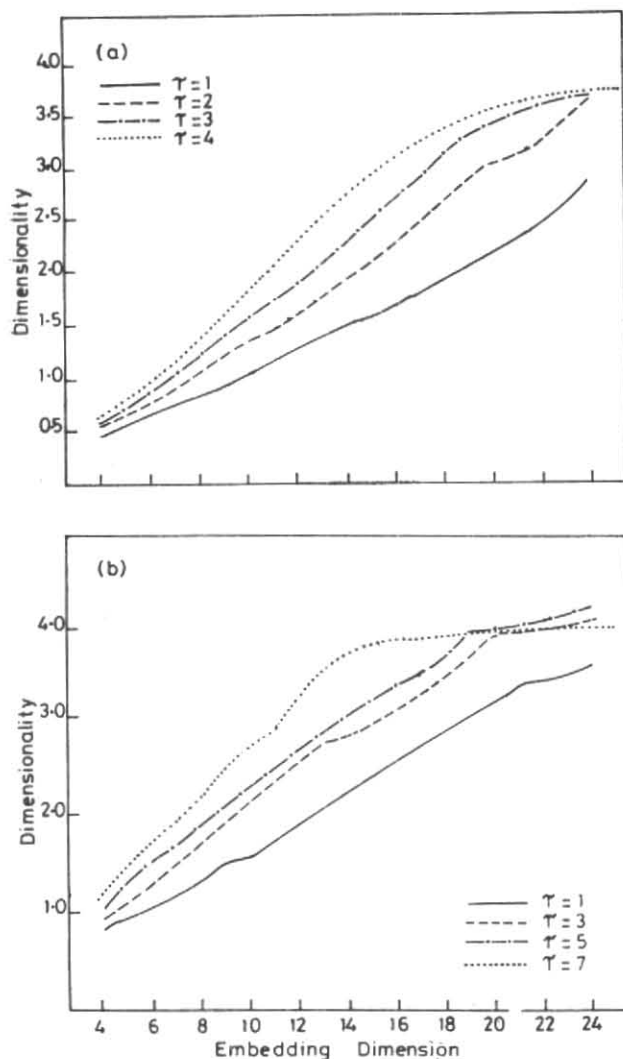
(Atkinson 1981). The data of each year was subjected to power spectrum analysis by adapting the method suggested by Blackman and Tukey, as given in WMO Technical Note No.79 (1966). The power spectrum was constructed with a maximum lag of 31 which is 1/3 of the frequency of the data of each year, viz., 92. The results of the red noise spectrum (Markov type persistence) is summarized in the Table 1. The year to year variability of strong and the weak periodicities suggests that the time series is chaotic.

4.2. Fractal dimension analysis

As a fairly long time series is required for the determination of fractal dimension, 1380 daily maximum temperature data points (15 years data of 92 values each) of MO as well as ACWC Madras have been considered independently. Since persistence in the daily maximum temperature is quite common, in order to ensure the linear independence of the data the series have been subjected to correlogram analysis for each year to identify the maximum lag at which the auto correlation co-efficient is insignificant. For this purpose the method suggested by Alan Pankartz (1983) has been employed. It was found that the persistency was seen upto lag 7 for MO as well as for ACWC maximum temperature series (see Table 1). However, when the anom-

aly series (actual-mean) was considered, persistency was restricted to lag 4 only. Hence the timeshift (τ) has been restricted to 7 in the case of original series and 4 in the case of anomaly series.

As there will be long time memory involved between data points of successive years (since the data points are separated by 365 days, i.e., between the data set of one year to that of the next year), the distributive functions are obtained for each year and composited thereafter for the entire period (see Fraedrich 1986). Fig.3 shows the typical graph of $\log C(l)$ against $\log (l)$ for $\tau=1$ in respect of MO Madras. The graph has a linear portion (slope) for various embedding dimension (m) and this slope has been found out analytically by fitting a linear best fit curve by the method of least square. The slope thus obtained for each embedding dimension is called the dimensionality (d). Figs. 4(a & b) show the gradual increase of dimensionality with the increase of embedding dimension. The asymptotic or saturating value of dimensionality is the fractal dimension of the attractor. However, as this sort of observed time series is expected to contain some environmental noise, perfect asymptotic value of different dimensionalities are not possible many a time. In order to get the asymptotic value, the pentad average of the dimensionality (d) for various embedding



Figs.4(a&b). Graph of dimensionality vs embedding space for (a) MO Chennai (actual maximum temperature series) and (b) ACWC Chennai (maximum temperature anomaly series)

dimensions (m) are computed $\left[\sum_{i=-2}^{+2} d(m+i)/5 \text{ for } m=4 \text{ to } 24 \right]$

to smooth the values. The slope is then worked out and the near zero slope (say less than 10 degrees) is identified as the asymptotic value and the corresponding dimensionality is the fractal dimension (D). The integer obtained by rounding off the fractal dimension to the higher side indicates the minimum number of independent variables necessary to model the dynamics of the attractor, while the maximum number of variables that are sufficient to model is that embedding dimension (m) beyond which the dimensionality is asymptotic or saturated (Fraedrich 1986). The fractal dimension of chaotic attractor of maximum temperature series of MO Madras is 3.9, while that of ACWC Madras is 3.5 suggesting a minimum of 4 independent parameters are required to understand and model the system. The embedding dimension at which the saturation is reached (*i.e.*,

where the dimensionality obtained for subsequent embedding dimension is asymptotic) is the upper bound (degrees of freedom) of parameters required to model the system in this case 19.

5. Results and discussion

5.1. Limitations

The theory of chaos has the basic assumption that the transients have died out and the motion has reached the attractor. But in many real world problems this assumption may not be realised. Since the observables may contain the environmental (statistical) noise which may not be identified had eliminated in real problems in which case the advantage of deterministic process over stochastic process will be lost.

5.2. Scope for future work

As of now, there is no unified procedure to identify the independent parameters though the lowerbound and upperbound of such parameters needed to model are brought out by the theory of deterministic chaos. Fraedrich (1986) offered physical interpretation for the dimensionality of weather (fractal dimension of pressure in his study was 3.5-3.9) as cyclones (short period disturbances), troughs of slow moving waves and an index cycle of these two forcings. In a similar way the dimensionality of maximum temperature could be explained by means of sea breeze circulation (their extent, strength and intensity), airmass advection and an interaction of these two disturbances in an index cycle of reduced or enhanced synoptic activities. Chatterjee and Yilmaz (1992) suggested that the variables can be any one of the past observables of the same timeseries or it could be some other related variable. Based on these lines, identification of parameters are being carried out to forecast the maximum temperature over Madras.

6. Conclusion

The minimum number of variables that are necessary to model the maximum temperature over Madras is 4 and the maximum (upperbound) number of sufficient variables is 19.

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