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An analytical model for mountain wave in stratified atmosphere

P. KUMAR*

Faculty of Meteorology, Air Force Admn. College, Coimbatore

M.P. SINGH

Centre for Atmospheric Sciences, Indian Institute of Technology, New Delhi

AVM. N. NATARAJAN

Directorate of Meteorology, Air HQ(VB), New Delhi

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सार — कश्मीर घाटी में पीर पंजाल पर्वत श्रेणियों के कारण उठने वाली पर्वतीय तरंगों के संबंध में वास्तविक समय पूर्वानुमान के लिए एक द्विआयामी विश्लेषक कम्प्यूटर मॉडल विकसित किया गया है। उच्चतर स्तर पर वास्तविक शून्येतर मानों तथा निम्न स्तर के चरघातांकी ह्रासमान मानों के साथ L^2 प्रोफाइल के अनुकरण को तैयार किया गया है। डूस (1961) के विपरीत वर्तमान समाधान तरंग संख्या (k) के मान को बाधित नहीं करता है। सात मामलों में उपग्रह द्वारा प्रेक्षित की गई तरंगों से तथा एक मामले में वास्तविक वायुयान रिपोर्ट से मॉडल की मान्यता की जांच की गई है।

ABSTRACT. An analytical, two-dimensional computer model has been developed for real time prediction of mountain wave due to Pirpanjal mountains over Kashmir valley. Simulation of the L^2 profile has been made with realistic, non-zero values at higher levels and exponentially decreasing values at lower levels. Unlike Doos (1961), present solution has no restriction on the value of wave number (k). Validity of the model has been tested with the satellite observed waves in seven cases and actual aircraft report in one case.

Key words—Atmospheric, Mountain, Wavenumber, Waves.

1. Introduction

The problem of airflow over mountainous regions has been studied intensively for the past several years by using linear and nonlinear models. Lyre (1943) and Queney (1948) probably, provided the starting point for these investigations. They obtained some observed features of airflow over mountains in a simple two-dimensional, single layer model. Scorer (1949) used an atmosphere stratified in two layers. Although this representation of atmosphere is rather simplified, the results were illuminating and gave rise to extensive studies of two or more layer models. Scorer's model represented constant L^2 (Scorer's parameter) profile in the two layers with discontinuity at the interface. Since then a number of attempts were made to simulate the actual atmospheric profile more realistically.

Although linearized theory of internal gravity waves due to orography is strictly applicable to shallow mountains and small amplitude waves only, it has been widely used to study large variety of mountain shapes and atmospheric profiles, to obtain a fairly good idea of airflow pattern (Doos 1961, Vergeiner 1971, Klemp and Lilly 1975). Philips and Brown (1983) has recently carried out experimental validation of the waves predicted by the linearized theory and actual aircraft observations. They found that linearized theory is good enough in predicting the wave-length. Amplitude of the wave, however, was noticed to be slightly more than what was predicted by the linearized model. Over Indian region Sarker (1965) studied the waves generated due to Western Ghats.

Over Jammu & Kashmir region, Pirpanjal ranges, having near north-south orientation form a favourable barrier for the generally prevailing westerly winds in this region.

Earlier attempts to simulate the Scorer's profile as a continuous model may be broadly classified into two categories - an exponentially decreasing single layer model (Foldvik and Palm 1959, Sarker 1965) and a multilayer exponential model with appropriate matching conditions for vertical velocity and its vertical gradient (Danielson and

Communication address: Retd. Wg. Cdr. P. Kumar, C-7/159, Senpura, Chetgunj-ward, Varanasi-221 001.

Bleck 1970, Palm and Foldvik 1960). Although the later approach is quite close to reality, the relevant expressions become increasingly cumbersome with increasing number of layers. While the former approach is simpler, exponentially decreasing L^2 - profile not only restricts its applicability to limited number of cases but also does not represent true atmospheric conditions at higher levels. Doos (1961) attempted to study the realistic profile with constant non-zero values (say L^2_∞) at higher levels but neglected the contribution to the Fourier integral for values of wavenumbers less than L^2_∞. This paper illustrates a method to solve the lee-wave equation with similar stratification but for the entire wavelength spectrum for airflow over the Pirpanjal ranges.

2. Formulation of the problem and solution

Assuming that the mountain is shallow, wave amplitude is small and the airflow is inviscid and laminar, the governing equation for the vertical component of the velocity, under Boussinesq approximation, can be expressed as (Scorer 1949)

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + L^2(z)\, w = 0 \qquad (1)$$

where for

$$\beta = \frac{1}{\theta}\frac{d\theta}{dz}; \quad S = \frac{1}{\rho}\frac{\partial \rho}{\partial z}$$

$$L^2(z) = \left(\frac{g\beta}{U^2} - \frac{1}{U}\frac{d^2U}{dz^2} + \frac{S}{U}\frac{dU}{dz} - \frac{S^2}{4} + \frac{1}{2}\frac{ds}{dz}\right) \qquad (2)$$

Boundary Conditions

(*i*) Along the mountain [$z = \xi(x)$] the continuity of the normal component of velocity leads to

$$\frac{W}{U} = \left.\frac{d\xi}{dx}\right|_{z=0} \qquad (3a)$$

(*ii*) For large values of z, we will take the more realistic radiation condition implying the upward propagation of energy as $z \to \infty$

$$w \propto \exp(i\nu z).\ \nu = \text{a constant} \qquad (3b)$$

To represent the stratification of the atmosphere, we consider a single layer profile given by

$$L^2 = L^2_\infty + L^2_0\, e^{\alpha z},\ \alpha < 0 \qquad (4)$$

Here, L_∞, L_0 and α are determined by fitting in the observed atmospheric data. Applying Fourier transform to Eqn. (1), we obtain

$$\tilde{w}_{zz} + (L^2 - k^2)\,\tilde{w} = 0 \qquad (5)$$

where,

$$\tilde{w}(k,z) = \frac{1}{\pi}\int_{-\infty}^{\infty} W(x,z)\, e^{-ikx}\, dx$$

and $w(x,z) = \int_0^{\infty} \tilde{w}(k,z)\, e^{ikx}\, dk$

Substituting Eqn.(4) in Eqn.(5) we get,

$$\tilde{w}_{zz} + (L^2_\infty + L^2_0\, e^{\alpha z} - k^2)\tilde{w} = 0 \qquad (6)$$

Eqn.(6) can be reduced to the more familiar form

$$\frac{d^2\tilde{w}}{d\mu^2} + \frac{1}{\mu}\frac{d\tilde{w}}{d\mu} + \left(1 - \frac{\nu^2}{\mu^2}\right)\tilde{w} = 0 \qquad (7)$$

where

$$\mu = \frac{2L_0\, e^{\alpha z/2}}{\alpha}$$

and

$$\nu = i\,\frac{2(L^2_\infty - k^2)^{1/2}}{\alpha} = ip \text{(say)} \qquad (8)$$

The solution of Eqn.(7) satisfying the radiation condition is (cf. Appendix I)

$$\tilde{w} = A J_{-\nu}(\mu) \qquad (9)$$

Where, A is an arbitrary function of the wavenumber k. The mountain profile is taken in the form

$$\xi(x) = \frac{Ha^2}{a^2 + x^2} + \frac{d}{dx}\left(\frac{bc^2}{c^2 + x^2}\right) + \frac{df^2}{f^2 + (x - 36.0)^2} \qquad (10)$$

which represents an analytical expression for the actual ground profile between Peshawar (34.01°N/71.35°E) and Srinagar (34.05°N/74.58°E) across Pirpanjal mountains. Values of the parameters are computed by fitting in Eqn.(10) with the actual topography (Fig.1) by using the method of least square (employing Marquardt numerical technique-Kuestner and Joe 1973) and are given as

$$H = 1.9;\ a = 8.6019796;\ b = 39.353073;$$
$$c = 49.038303;\ d = 1.8835262;\ f = 178.43532; \qquad (11)$$

The mountain profile Eqn.(10) comprises of linear combination of bell-shaped functions (symmetrical with respect to the vertical axis) and its gradient function (asymmetrical with respect to the vertical axis). Addition of two or more of these types of functions would be a convenient analytical tool to formulate complicated ground profile into a simple mathematical model.

Fourier transform of Eqn.(10) gives

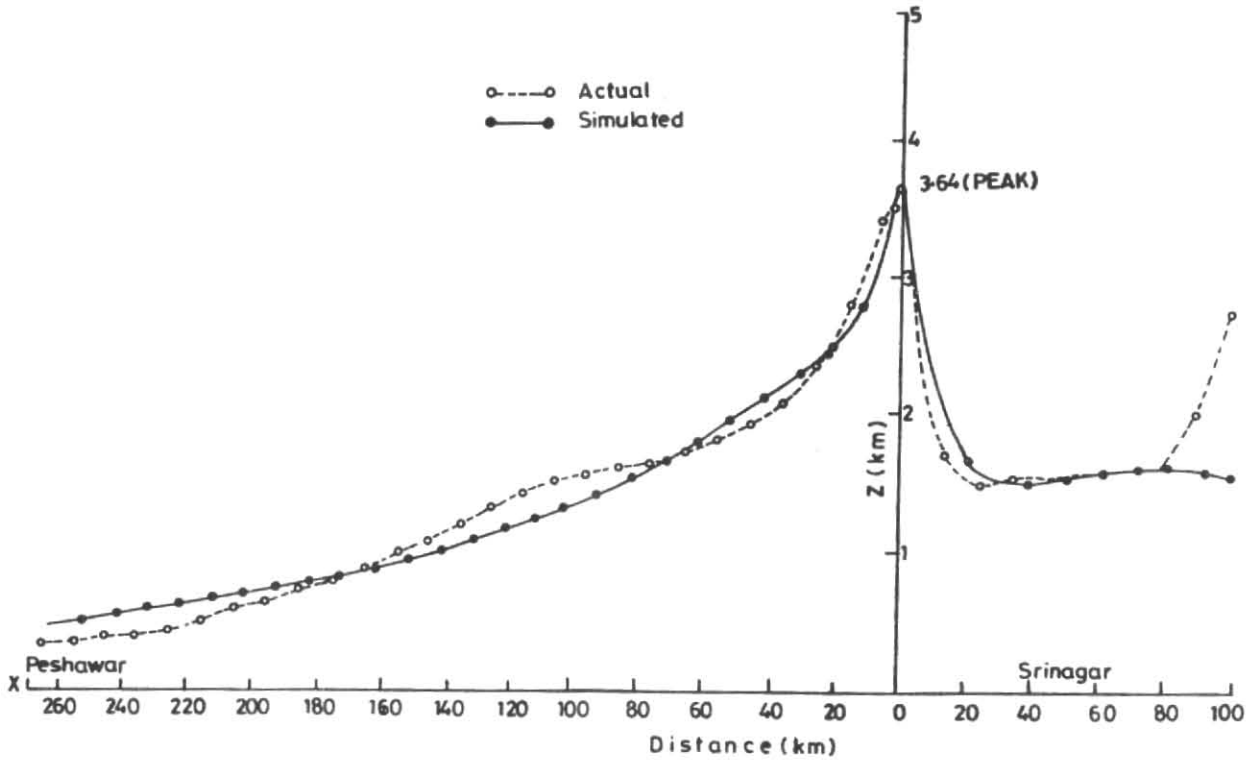

Fig.1. Profile of E-W axis of Pirapanjal mountain cross section

$$\xi(k) = Hae^{-a|k|} + ikbe^{e^{-e|k|}} + dfe^{-(f|k| + i36k)} \tag{12}$$

and the boundary condition in Eqn.(3a) takes the form

$$\frac{\tilde{w}}{U} = ik\,\xi(k)\Big|_{z=o} \tag{13}$$

Using Eqn.(13) in Eqn.(9), we obtain

$$\tilde{w} = \frac{ik\,\xi(k)\,U(0)}{J_{-ip}(\mu_0)}\,J_{-ip}(\xi) \tag{14}$$

Where, $\mu\big|_{z=0} = \mu_0$

Taking inverse Fourier transform of Eqn.(14) we get,

$$w = R_e\int_0^\infty \frac{ik\xi(k)\,U(0)}{J_{-ip}(\mu_0)}\,J_{ip}(\mu)\,e^{ikx}\,dk \tag{15}$$

where

$$p = \frac{2\,(L_\infty^2 - k^2)^{1/2}}{|\infty|}$$

According to Coulomb (1936) the zeroes of $J_{ip}(\mu_o)$ are regarded as real and simple for positive values of the argument μ_o. The integral at Eqn.(15) can now be evaluated by using the method of residues.

Applying Kelvin's monotony condition, we get,

(*i*) For $x > o$

$$w = Re\int_0^\infty \frac{1}{J_{-is}(\mu_o)} -ik\,[1.9ae^{-iak} - kbee^{-iek} + dfe^{-i\,(f+36.0i)\,k}]\,U(0)\,J_{-is}(\mu)\,e^{-kx}\,dk + 2\pi i\,(Residue) \tag{16a}$$

(*ii*) For $x < o$

$$w = Re\int_0^\infty \frac{1}{J_{-is_1}(\mu_o)} \{-ik\,(1.9\,ae^{iak} + kbee^{iek} + dfe^{i\,(f+36.0i)\,k})\,U(0)\,J_{-is_1}(\mu)\,e^{kx}\}dk \tag{16b}$$

In the above, $p = s$ along the positive half of the imaginary axis and $p = s_1$ along the negative part of the imaginary axis in the complex k-plane.

$$\text{Residue} = \sum_{p=1}^{N} \frac{1}{\frac{d}{dk}\,(J_{-ip}(\mu o))_{k=k_{P_N}}}$$

$$\times \{ik_{P_N}\,(1.9ae^{-ak_{P_N}} + ik_{P_N}\,bee^{-ek_{P_N}} + d\,fe^{-(f+36.0i)\,k_{P_N}})\,U_{(0)}\,J_{-iP_N}(\mu)\,e^{ik_{P_N}} \tag{17}$$

TABLE 1

Date	L_∞^2 (km^{-2})	L_0^2 (km^{-2})	α (km^{-1})	k_p	Computed wavelength	Observed wavelength	Remarks
4 Jan 1984	0.1541E-02	21529.249	2.4035	0.57097031 2.7940077 4.8228662	11.0 2.2 1.3	11.0	
16 Jun 1984	0.01696	3108.59	1.957	0.3932 2.1516 3.7370	15.9 2.9 1.6	16.0	
14 Aug 1985	0.1851	4254.0	2.0	0.4 1.82	14.1 3.4	15.00	
23 Aug 1985	0.27317804 E+08			0.69348182 2.1181984 4.7226503	9.0 2.9 1.3		
5 Sep 1985	0.08081	100.00	1.4559	0.4871 1.0007 2.0176	12.8 6.2 3.1	12.0	
14 Sep 1985	0.3581	1312.7396	1.92	0.59846839 1.8339 3.2645703	10.4 3.4 1.9	11.0	
24 Sep 1985	0.0038	54467.23	2.489	0.5093 1.9274 4.1882	12.3 3.2 1.5	12.0	
7 Dec 1985	0.1611	2408.8	1.5	0.4 1.39 2.8	15.4 4.5 2.2	13.00	Aircraft report turbulence at 7.5km level

where

$P_N = P_{k=k_{PN}}$ and K_{P_N} are the real zeros of $J_{-ip}(\mu_0)$

The vertical velocity $w(x,z)$ is given by Eqn.(16) and Eqn.(17).

3. Comparison of the predicted value with observed data

For the purpose of validation of our results we selected AVHRR pictures from NOAA satellite of lee-waves over Kashmir valley, which is situated on the leeward side of Pirpanjal ranges. This range is about 2900 km long (aligned in the 340° - 160° direction) and 50 km wide with average height of 3.6 km. Viewing from the plains of Punjab this is the first mountain range we encounter in the North/North-west. Peshawar is situated west of this range at about 260 km from the peak and at 0.36 km amsl. East of Pirpanjal is Kashmir valley at 1.5 km amsl. This valley is about 70 km wide. Further east of this valley is the Great Himalayan ranges.

Fig.1 shows the two-dimensional topography along the Peshawar- Srinagar axis up to Srinagar only. With this as the lower boundary, our solution will be valid only up to the Srinagar valley. Actual aircraft reports indicate that turbulence is normally reported at the height of 6-8 km. This level is much above the average height of the other ranges in the region (average height of the Greater Himalayan barrier is 5 km). Effect of the Himalayan barrier, therefore, may be assumed to cause no significant difference on the high level turbulence or wave profiles over Srinagar valley region. Hence, in this study we have neglected the effect of the Greater Himalayan ranges on the lee-waves over the Srinagar valley. Wind and temperature distribution representing the upstream conditions were obtained from the upper air sounding data of Peshawar.

Computer program was developed for the computation of L^2 from actual data. The entire vertical region was divided into equal intervals of 500 meters each and the data for missing levels were interpolated linearly. L^2 values were computed with centred difference for intermediate levels and forward and backward difference for the bottom and top levels respectively. Evaluation of L_∞ , L_0 and α values were done by using the Marduardt method of least square approximation. This is an extension of Gauss-Newton method to allow for the convergence with relatively poor starting guesses for unknown coefficients L_∞, L_0 and α. In general, the steepest descent procedure also converges for poor starting values, but requires a longer solution time in comparison to the Marquardt method. An example of L^2 profile over Peshawar is shown in Fig.2. Table 1 enlists the dates on which lee-waves were observed over Kashmir valley. Last two columns show the observed and the computed wavelengths.

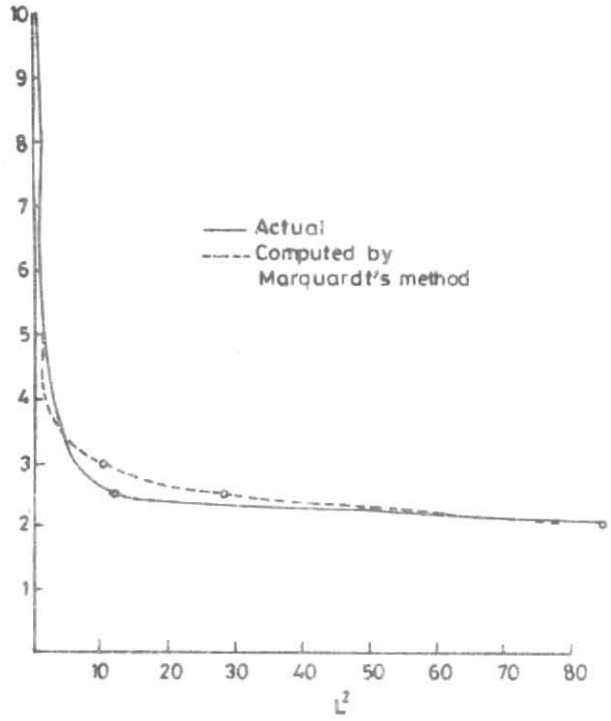

Fig.2. L^2-profile over Peshawar at 0000 UTC on 14 August 1985

Computed values reasonably match with the observed ones in most of the cases. On 7 December 1985 the satellite observation was corroborated with an aircraft report of moderate turbulences by one of the aircrafts flying at 7.5 km level over Kashmir region at about 0600 UTC. Minor variations in the computed and the observed wavelength on some of these days may be attributed to the variation of the atmospheric conditions from the time of upper air sounding to the time of NOAA satellite pictures. For the sake of convenience, values of L_0^2 were calculated at 0.5 km amsl above Peshawar rather than at 0.36 km amsl (which is the elevation of Peshawar). This is not expected to materially affect our computation and results. It was found that wavenumber K_{P_N} is highly sensitive to the values of L_∞^2, and L_0^2. Hence, achieving the best Fit for L_∞^2, L_0^2 and α values with L^2 values, computed from observed data, is of great significance in our study. On 14 August 1985, unfortunately an easterly wind component was encountered upto 1.5 km level due to monsoon currents. Hence, μ_0 on this date was computed at the next higher level, *i.e.*, at 2.0 km to get the correct resonance wavelength, although the computed value of w is not expected to be reliable in this case. Very high L_0^2 value on 23 August 1985 may be attributed to light wind (not calm) prevailing on this day over Peshawar in lower levels.

4. Discussion

The solution for the vertical velocity and displacement, without any restriction on the value of k, is physically significant. Small wavenumber ($0 < k \leq L_\infty$) give rise to long waves commonly known as hydrostatic mountain waves ($\geq$ 30 km). (Philips and Brown 1983), Doos (1961), while attempting to explain the behaviour of the solution in the domain ($0 < k < L_\infty$) neglected the contribution to the Fourier integral for values of k in this interval since he considered their contribution to be very small.

We note from Eqns.(16a & b) that the contribution is proportional to the term e^{-kx}. For small values of k and x, the value of the integral cannot be considered small enough to be neglected. It will decay exponentially with horizontal distance x from the mountain top. Vertical propagation of these waves, far deeper into the atmosphere, even up to 25-30 km in the stratosphere, has been observed since long, as lenticular and nacreous (mother of pearl) clouds (Alaka 1958, Queney *et al.* 1960). Energy propagation to these heights is well explained by qualitatively applying the radiation condition as shown in the Appendix I. Behaviour of these long waves in lower levels have been analysed by various workers (Klemp and Lilly 1975, Smith 1977), AVHRR pictures selected for the validation purposes. Over Jammu & Kashmir region, in the present study, indicate only lee-waves. An experimental case study, however, for observed hydrostatic mountain wave may justify the above view on small wavenumbers.

Lower boundary in Eqn.(10) provides a simple mathematical tool for formulating a function for rugged hilly terrain as in Jammu & Kashmir region. Marquardt's method of least square approximation not only provides the values of constants for such functions, but also provides us the best fit values of L_∞^2, L_0^2 and α for L^2 profile computed from the observed data. This method of approximation is also useful for computing the resonant wavelength even in the case when low level winds are negative within the shallow layers from the ground as on 14 August 1985.

Appendix I

Application of radiation condition

The general solution of Eqn.(7) for various values of k is

$$w = A\,J_{-ip}(\mu) + BY_{-ip}(\mu) \text{ for } k \geq L_\infty \qquad \text{(AI.1)}$$

$$w = C\,J_{-ip}(\mu) + D\,J_{ip}(\mu) \text{ for } k < L_\infty \qquad \text{(AI.2)}$$

If $k \geq L_\infty$, ip is real and in that case as $z \to \infty$ or $\mu \to 0$, $Y_{-i_p}(\mu) \to -\infty$. Hence we must drop the second term in (AI.1) to fulfil the radiation condition. When $k < L_\infty$ solution (AI.2) is given in terms of Bessel functions with imaginary order. Doos (1961), using Boole's (1844) expansion, erroneously applied the rigid bad condition and found that the solution is identically zero for $k < L_\infty$. He finally neglected the contribution to the Fourier integral for values of k in the interval $0 < k < L_\infty$. Behaviour of Bessel function with imaginary argument, however, is very little discussed in the literature. Therefore the following approach is adopted to determine the behaviour of the function when the argument tends to zero :

$$\underset{\mu \to 0}{\text{Limit}}\ J_{ip}(\mu) = \underset{\mu \to o}{Limit}$$

$$\left(\frac{\mu}{2}\right)^{ip} \sum_{r=0}^{\infty} \frac{(-1)^r\ (\mu/2)^{2r}}{(ip + r + 1)\, L^r} \approx \left(\frac{\mu}{2}\right)^{ip} \frac{1}{(ip+1)}$$

$$= \underset{\mu \to 0}{\text{Limit}} \left\{\frac{1}{(ip+1)}\right\} e^{ip \log (\mu/2)}$$

$$= \text{Limit (Constant)}\ e^{ip \log(\mu/2)} \qquad \text{(A1.3)}$$

As $\mu \to 0$. $\log(\mu/2) \to -\infty$, $p \log(\mu/2) < 0$ as $\mu \to 0$. This indicates that $J_{ip}(\mu)$ represents a downward travelling wave. Similarly we can show that $J_{ip}(\mu)$ represents an upward travelling wave. Hence in (A.I.2) the second term is to be dropped. *i.e.*, $D = 0$. to satisfy the radiation condition. Thus, for both the cases (when $k \geq L_\infty$ and $k < L_\infty$), the uniform solution is

$$w = A\, J_{-ip}(\mu) \qquad \text{(AI.4)}$$

where A is an arbitrary function of k.

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