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Fractal dimensions of chaotic attractors for monsoon rainfall of meteorological sub-divisions of India

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सार — ग्रासबर्गर और प्रेाकासिया एल्गोरिथम विधि (1983) का प्रयेाग करते हुए समूचे भारत और सभी 35 मौसम वैज्ञानिक उपक्षेत्रों के दक्षिण-पश्चिम मानसून के विचित्र प्रभावों से सहसम्बद्ध फ्रैक्टल परिमाण और प्रायद्वीपीय भारत के 4 मौसम वैज्ञानिक उपक्षेत्रों की उत्तर-पूर्वी मानसून वर्षा का आकलन किया गया है । फ्रैक्टल परिमाण बहुत से पैरामीटरों की प्रारम्भिक सूचना देते हैं जो परिवर्तनात्मक मानसून प्रणाली को प्रभावित करने वाली गतिकी को समझने के लिए आवश्यक होती है । फ्रैक्टल परिमाण 2.9 और 7.1 के बीच भिन्नता दर्शाते हैं तथा 14 और 21 परिमाणों के मध्य संतृप्ति पाई जाती है । पांच उपक्षेत्रों में फ्रैक्टल परिमाण का पता नहीं लगाया जा सका है ।

ABSTRACT. The correlation fractal dimension of strange attractors of southwest monsoon rainfall of all the 35 Indian meteorological sub-divisions and India as a whole and north-east monsoon rainfall of 4 meteorological sub-divisions of peninsular India are estimated using the Grassberger and Procaccia algorithm (1983). The fractal dimensions provide us the primary information on the number of parameters that are required to understand the dynamics underlying the monsoon dynamic system. The fractal dimensions varied between 2.9 and 7.1 and the saturation occurred between 14 and 21 dimensions. In 5 sub-divisions the fractal dimension could not be determined.

Key words — Deterministic chaos, Dynamic systems, Fractal dimension, Strange attractor, Embedding dimensions, Phase space.

1. Introduction

The Southwest (SW) monsoon rainfall of India and its 35 meteorological sub-divisions has been widely studied based on data originating as far back as 1871. Reference can be made to Parthasarathy (1984), Mooley and Parthasarathy (1984), Subramanian *et al*. (1992). By and large the sub-divisional rainfall (SDR) and Indian monsoon rainfall (IMR) have been found to be homogeneous, free from persistence, normally distributed and possess periodicity of 2-3 years.

The last two decades have seen considerable interest generated in the study of dynamical systems and the associated theory of chaos (Gleick 1987 and Kaye 1993). A complex dynamical system based on several variables described by a set of non-linear equations eventually converges to a set called "attractor" whose dimension is less than that of the original state space and which may not be an integer. Systems with an attractor of non-integer dimension are said to be chaotic though they may be deterministic. A chaotic system is very sensitive to initial conditions, that is, trajectories originating closely may deviate considerably from each other after some time. Generally weather and climate systems do manifest such tendencies.

The attractor dimension of a weather or climate variable is the minimum number of independent variables that would be needed to model the system.

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Studies on this line have been conducted in India by Srivastava *et al.* (1994) for radio refractive index, Bhattacharya *et al.* (1995) for earthquake sequence. Satyan (1988) and Basu & Andharia (1992) have shown that the Indian monsoon rainfall (IMR) has fractal dimension of 5.1/5.4.

In this paper, we propose to determine the fractal dimension of southwest monsoon rainfall of all the 35 meteorological sub-divisions and also that of the northeast (NE) monsoon rainfall of four southern sub-divisions of India. Fractal dimension of IMR also will be discussed.

2. Theory and methodology

A smooth dynamical system defined by n independent variables $x_1, x_2, ..., x_n$ can be described by a set of n differential equations as follows :

$$\dot{x}_i = f_i\,(x_1, x_2, x_3, ..., x_n) \qquad (1)$$

$$(i = 1, 2, 3 \ldots n)$$

The vector $[x_1\,(t), x_2\,(t) ...., x_n\,(t)]$ describes the status of the system at time t. The Eqn. (1) can be reduced to an n-th order differential equation as follows:

$$x^{(n)} = f\{x, x', x'', x''', ..., x^{(n-1)}\} \qquad (2)$$

where, x is one of the variables $x_1, x_2 ......, x_n$ and that all the other variables have been eliminated by differentiation. Thus, the vector

$$x(t) = \{x(t), x'\,(t), x''\,(t) .... x^{(n-1)}\,(t)\} \qquad (3)$$

is treated as a single observation of the n-dimensional vector $[x_1\,(t), x_2\,(t), x_3\,(t), ..., x_n\,(t)]$ in the n-dimensional space. If the time series $x(t)$ is discrete we have,

$$x\,(t) = [x(t), x(t+\tau), x(t+2\tau), ..., x\{t+(m-1)\,\tau\}] \qquad (4)$$

in an m-dimensional phase space where τ is chosen so that the linear correlation between $x(t)$ and $x(t+\tau)$ is zero. Next, we define r_{ij} as the Euclidian distance between two vectors $x(t_i)$ and $x(t_j)$ and a function $N(l)$ as follows :

$$N(l) = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \theta\,(l - r_{ij})$$

where, θ is the Heaviside function defined by

$$\theta\,(a) = 0 \quad \text{if } a < 0$$
$$= 1 \quad \text{if } a > 0 \qquad (5)$$

and $K = N - (m - 1)$ is the maximum number of possible vector points with N data points in an m-dimensional phase space. $N(l)$ is the number of pairs separated by Euclidian distance less than l. We normalise $N(l)$ by dividing it by $k = (K)\,(K - 1)/2$ which is the total number of distinct pairs. Now, let $C(l) = N(l)/k$. Then,

$$C(l) = a \,.\, l^D$$

i.e. $$D = \underset{k \to 0}{lt}\ \{\log\ C(l)/\log\ (l)\} \qquad (6)$$

where, a is a constant. If the limit exists, D is defined as the correlation fractal dimension of the time series X (t). It may be pointed out here that there are other types of fractal dimensions also such as density and boundary fractal dimensions (Kaye 1993).

The value of D implies that the dynamical system after the transients die out reach its attractor whose dimension is D. The theory follows from celebrated papers of Whitney (1936), Hirsch (1976), Packard (1980), Takens (1981), Eckman (1981) and Fraedrich (1986). Chatterjee and Mustafa (1992) provide an elaborate description of chaos theory. If $D = \mathrm{I} + p$, where I is an integer and p is a fraction, then I + 1 variables would be needed to model the dynamical system under question. In practice, D is determined by plotting a graph of log $C(l)$ against log l for various values of m in Eqn. (4). The plot has a region of linearity and the slope of this portion is derived graphically or analytically. This could be called the embedded dimension $E(m)$. Subsequently a plot of m and $E(m)$ is drawn and the asymptotic value of $E(m)$ is taken as the fractal dimension. If $E(m)$ does not get asymptotic the fractal dimension does not exist and so the time series cannot be modelled by a finite number of variables. Then, the generating dynamical system behind the evolution of the series is said to be non-deterministic or randomly chaotic. The value of m at which $E(m)$ becomes asymptotic is also a parameter of interest as this is the maximum number of lags from which variables required to model the system would have to be chosen.

3. Data

The sub-divisional rainfall data for the period 1901-94 has been collected from the National Data

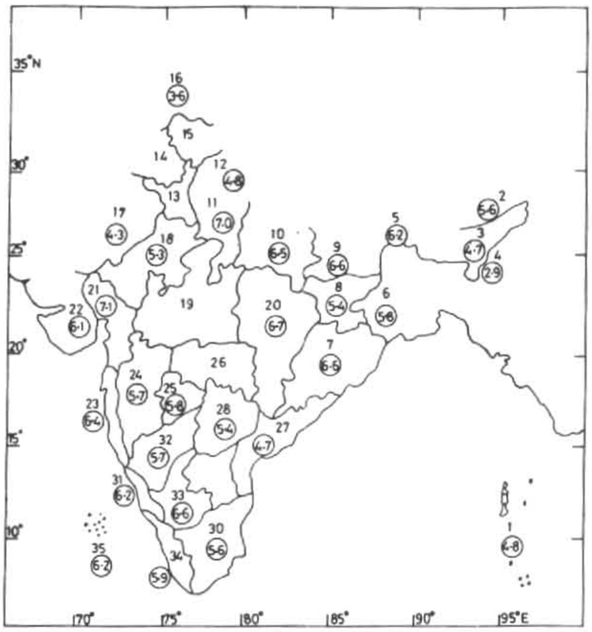

Fig. 1. Distribution of fractal dimension of southwest monsoon rainfall (in brackets)

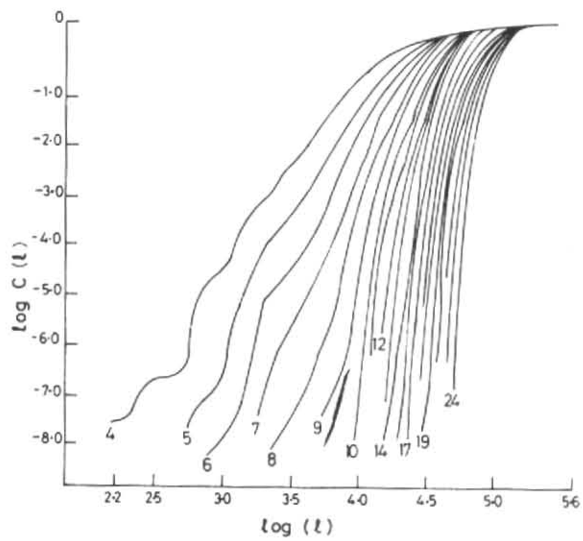

Fig. 2. Graph of log C (l) *vs.* log (l) for southwest monsoon rainfall of east Rajasthan

Centre, India Meteorological Department (IMD), Pune. The data for the year 1995 has been collected from the published records of IMD. Data of all the 35 sub-divisions was collected for the SW monsoon season. The IMR was computed from the SDR data of SW monsoon by area-averaging. Further data of 4 sub-divisions for the NE monsoon, where this season contributes substantially to the annual rainfall, was also collected. Thus, in all 40 rainfall series expressed in percentage departure from normal were subjected to further analysis. The geographical locations of various meteorological sub-divisions of India are shown in Fig. 1.

4. Computation and results

The SDR of various meteorological sub-divisions, when subjected to correlogram analysis, revealed that none of the auto-correlation co-efficient upto lag 20 was significant in almost all the sub-divisions confirming the linear independence of data points. In particular, the persistence was not observed upto lag 5 in all the cases. Hence, the timeshift (τ) has been taken as 1 and higher order of shifts were not tried. The computation of fractal dimension for the embedding dimensions varying from 2 to 24 was carried out for all sub-divisions. As can be expected this sort of observables may contain environmental (statistical) noise.

For the purpose of identifying the asymptotic value of the embedded dimensions the following analytical procedure was followed. The slope of the 5 points $[m+i-1, E(m+i-1)], i=1$ to 5 was computed for values of m = 2, 3, 4, ... When the graph of $[m, E(m)]$ is asymptotic, the above slopes are nearly zero and further the corresponding mean values of $E(m)$ do not differ substantially from one another. This stage was identified by a critical examination of slopes and the running means and the asymptotic value thus obtained was taken as the fractal dimension. The results are tabulated in Table 1 and presented in Fig. 1 for easy reference. The values of m at which saturation occurs is also given in Table 1. Fig. 2 shows the plot of log $C(l)$ *vs.* log (l) curve for a typical case of convergence in respect of east Rajasthan. The initial increase of the embedded dimensions with dimensions and the subsequent saturation in respect of east Rajasthan, Nagaland, Meghalaya, Manipur & Tripura and Haryana meteorological sub-divisions and IMR is shown in Fig. 3.

5. Discussions

From Fig. 1 and Table 1, it is seen that the fractal dimension of monsoon rainfall of 35 Indian meteorological sub-divisions varies between 2.9 and 7.1 with the saturation attained at the dimension of 14 to 21. Thus, the SDR could be modelled with 4

TABLE 1

Embedded dimensions and fractal dimensions of sub-divisional monsoon rainfall of India (based on computation of 1901-95 data)

Sub-division		Dimensions										Dimension	
No.	Name	2	4	6	8	10	12	14	16	18	20	Fractal	Saturation
		Embedded dimensions of southwest monsoon											
1	ANI	1.6	2.7	3.4	3.8	4.3	4.5	4.6	4.6	5.0	4.5	4.8	17
2	ARP	1.6	2.5	3.1	3.5	4.2	4.0	4.4	5.6	5.4	5.8	5.6	18
3	A&M	1.4	2.4	3.2	3.5	3.7	4.0	4.9	4.9	4.0	4.8	4.7	16
4	NMMT	1.3	2.0	2.6	2.3	2.5	2.6	2.6	2.8	2.9	2.9	2.9	17
5	SHWB	1.5	2.7	3.7	4.2	4.7	5.2	5.1	5.7	6.2	6.4	6.2	19
6	GWB	1.7	3.0	3.5	3.9	4.6	5.1	4.9	5.3	6.1	6.1	5.8	18
7	ORS	1.6	3.0	3.6	4.3	5.2	6.0	6.4	5.9	6.7	6.7	6.6	18
8	BPU	1.7	2.9	3.5	4.2	4.9	5.2	5.2	5.2	5.4	5.6	5.4	18
9	BPN	1.6	2.7	3.5	4.0	4.6	5.0	4.8	5.2	6.7	6.8	6.6	21
10	EUP	1.8	2.9	3.4	4.4	4.8	5.3	6.0	6.6	5.9	5.9	6.5	17
11	WUP	1.7	2.9	4.2	5.1	5.6	5.9	6.3	6.3	6.7	7.5	6.9	17
12	HUP	1.6	2.9	3.3	3.9	4.6	5.4	4.7	4.5	4.9	5.1	4.8	15
13	HAR	1.6	3.0	3.8	4.8	4.3	5.0	5.5	5.8	6.8	6.7	–	–
14	PJB	1.7	2.7	3.4	4.6	5.4	5.7	6.7	5.7	6.1	6.5	–	–
15	HP	1.6	2.8	3.6	4.6	4.9	4.4	4.7	5.8	5.6	6.8	–	–
16	J&K	1.6	2.3	2.8	3.0	3.1	3.9	3.9	4.1	3.4	3.5	3.6	15
17	WR	1.7	2.7	3.5	3.7	3.3	4.3	4.2	4.3	4.5	5.3	4.3	17
18	ER	1.7	2.8	3.5	4.5	4.2	3.9	5.7	5.2	5.2	5.4	5.3	18
19	WMP	1.6	2.8	3.8	4.3	5.5	6.8	6.8	6.7	6.8	7.2	–	–
20	EMP	1.5	3.0	3.5	4.2	4.7	5.0	5.8	6.2	6.7	6.9	6.7	20
21	GUJ	2.0	2.9	4.2	4.8	5.8	5.2	7.1	6.3	7.6	7.7	7.1	17
22	S&K	1.9	3.1	4.2	4.4	5.7	6.2	6.4	6.1	6.3	8.5	6.1	17
23	K&G	1.6	2.6	3.5	4.1	4.8	5.6	6.5	5.2	6.5	7.3	6.4	18
24	MM	1.6	2.9	3.5	4.2	4.9	5.0	4.7	5.1	5.6	6.1	5.7	20
25	MWD	1.7	2.9	3.8	4.0	4.4	5.1	5.0	6.1	4.9	6.0	5.8	18
26	VID	1.7	2.8	3.9	5.0	5.4	5.4	5.8	6.7	7.3	8.0	–	–
27	CAP	1.7	3.0	3.2	3.8	4.3	4.3	4.9	4.6	5.2	5.2	4.7	16
28	TLG	1.6	3.1	3.4	4.4	4.8	4.9	5.4	5.1	5.4	6.1	5.4	14
29	RYL	1.6	2.6	3.1	4.0	4.7	5.2	5.8	6.0	6.1	5.5	5.9	16
30	TN	1.6	2.7	3.6	4.2	4.9	3.8	4.7	5.4	6.0	5.8	5.6	19
31	CK	1.6	2.9	3.4	4.1	4.5	5.0	5.6	5.7	6.9	6.8	6.2	17

TABLE 1 (*Contd.*)

Sub-division		Dimensions										Dimension	
No.	Name	2	4	6	8	10	12	14	16	18	20	Fractal	Saturation
32	NIK	1.6	3.1	4.0	4.4	4.8	5.3	5.3	5.5	5.4	6.6	5.7	17
33	SIK	1.7	2.9	3.6	4.0	5.2	5.2	6.0	6.7	7.4	7.0	6.6	17
34	KER	1.7	3.0	3.5	3.9	4.4	5.2	4.7	5.4	5.4	6.5	5.9	18
35	LAK	1.7	2.7	3.8	4.5	4.4	5.0	5.7	6.3	5.7	6.1	6.2	17
All India		1.5	2.7	3.5	4.7	4.9	4.4	4.9	6.1	6.7	6.6	6.7	21
Embedded dimensions of northeast monsoon													
27	CAP	1.7	3.0	4.2	4.3	5.4	5.6	6.6	6.9	6.5	7.0	6.8	17
29	RYL	1.6	3.1	4.0	4.9	5.0	5.5	7.1	6.6	5.6	7.1	7.2	15
30	TN	1.7	2.9	3.7	4.5	4.4	4.8	5.6	6.2	6.1	6.8	–	–
34	KER	1.9	3.1	3.9	4.8	5.4	5.0	5.5	4.8	7.1	6.3	6.5	19

Notes : (1) Saturation Dimension is the dimension at which Embedded Dimension becomes asymptotic (i.e. reaches the fractal dimension) (2) Blank denotes no saturation

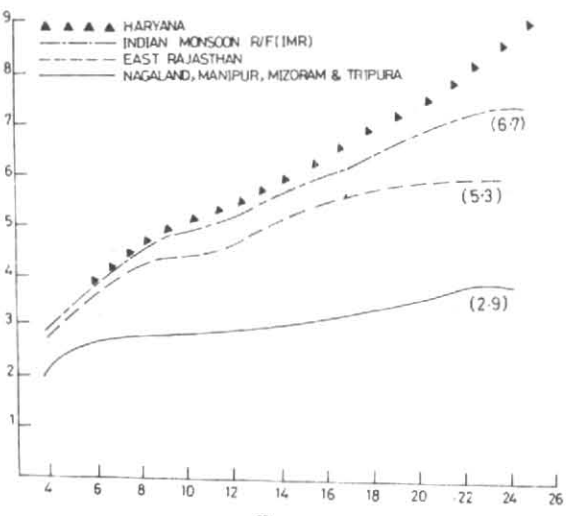

Fig. 3. Graph of embedded dimension E(m) *vs.* dimension(m)

to 7 variables. The fractal dimension is slightly lower in the northeast region whereas in 5 sub-divisions, 3 located in extreme north and 2 in central India, it does not exist. In such cases the computation was extended upto 30 dimensions and the results obtained confirmed non-saturation. This suggests that with the sample size employed in the study it may not be possible to model the dynamics underlying the SW monsoon rainfall of the above 5 sub-divisions. The IMR has a dimension of 6.7 slightly higher than the value of 5.1 obtained by Satyan (1988) and 5.4 obtained by Basu and Andharia (1992).

The data of IMR used in this study is derived from data of all the 35 sub-divisions and is based on observations of thousands of raingauge stations whereas in the other studies data of 29 sub-divisions based on nearly 300 stations only has been used. The NE monsoon rainfall manifests fractal dimension of 6-8 except for Tamil Nadu for which there was no saturation of the embedded dimension.

The fractal dimensions do provide us information about the deterministic nature of the system. However, any direct application towards the predictability of systems for the monsoon rainfall emerging from the above analysis appears doubtful as of now. Firstly, due to noise in the data, for a time series Eqn. (1) could be re-written as follows :

$$x_{t+1} = f(x_t, x_{t-1}, x_{t-2}, ..., x_{t-L}) + \varepsilon_t \qquad (7)$$

where, L is the lag and ε_t is a random component of environmental noise (Chatterjee and Mustafa 1992). If it is large or stochastic, it may be difficult to separate it from the deterministic portion. Secondly, the definition of attractor has the underlying assumption that transients have died out and the motion has reached the attractor. It is perhaps unlikely that such a stage will ever be realised in respect of a parameter such as monsoon rainfall.

Despite above, the fractal dimensions derived in this paper supply an interesting information, *viz*., how various time series could be characterised based on the Euclidian distance between the innumerable pairs of points in the n-dimensional phase space. The difference between random process and deterministic process also clearly emerges. Except for weak 2-3 years' periodicity, the SDR and IMR series are all linearly random with insignificant auto-correlation co-efficients. However, as we have seen several such series are deterministic and could be modelled based on the past values, though relations relating the variables are unknown and the modelling is based on the assumption that the motion has reached the attractor - *a* situation for which we have no means to verify.

6. Conclusion

The fractal dimensions of strange attractors of SW monsoon rainfall of India and its 35 meteorological sub-divisions were estimated. This information gives an idea on the lower-bound and upper-bound on the number of variables/parameters required to model the dynamics of the system. Also, the non-existence of the strange attractor (revealed by infinite fractal dimension) for 5 meteorological sub-divisions suggests that the dynamics of such systems is randomly chaotic. The fractal dimensions of NE monsoon rainfall of 4 sub-divisions in southern peninsular India have also been estimated.

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