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Classification and characteristics ofabrupt change based on the Lorenz equation

CHAOJIU DA, TAICHEN FENG**,***, BINGLU SHEN*# and JIAN SONG*##

School of Mathematics and Computer Science Institute, Northwest Minzu University,

Lanzhou, Gansu, 730000, China

**Institute of Applied Mathematics and Astronomical Calendar, Northwest Minzu University,*

Lanzhou, Gansu, 730000, China

***School of Atmospheric Sciences and Key Laboratory of Tropical Atmosphere-Ocean System Ministry of Education,*

Sun Yat-sen University, Zhuhai, 519082, China

****Southern Marine Science and Engineering Guangdong Laboratory, Zhuhai, 519082, China*

**#Gansu Weather Modification Office, Lanzhou, Gansu, 730020, China*

**##College of Science, Inner Mongolia University of Technology, Hohhot, Inner Mongolia, 010051, China*

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e mail : dmwsj@163.com

सार — **इस शोध पत्र में, एक गतिकीय प्रणाली में फोर्सिंग टर्म द्वारा प्रेरित अचानक परिवर्तन पर प्रारंभिक सैद्धांतिक शोध के बारे में बताया गया है। लोरेंज समीकरण प्रक्षेप वक्र को अनुसंधान के रूप में लेते हुए,भिन्न-भिन्न पल्स फोर्सिंग शब्दों के प्रक्षेप वक्र प्रतिक्रिया का अध्ययन अंतर समीकरणों और संख्यात्मक तरीकों की स्थिरता प्रमेय के आधार पर किया गया है। एक गतिकीय प्रणाली के परिप्रेक्ष्य से, अचानक परिवर्तन को आंतरिक या बाह्य रूप में वर्गीकृत किया जा सकता है। पहले वाला एट्रैक्टर अंदर के प्रक्षेप पथ के आत्म समायोजन को दर्शाता है, जबकि बाद वाला एट्रैक्टर से विचलन में प्रक्षेप पथ के विचित्र व्यवहार को दर्शाता है। यह वर्गीकरण अचानक वायुमंडलीय परिवर्तन की विभिन्न अभिव्यक्तियों के भौतिक तंत्र को समझने में मदद करता है। पल्स फ़ोर्सिंग टर्म की विभिन्न तीव्रताओं और अवधियों के लिए, जिन्हें क्रमशः एक आयताकार तरंग के परिमाण और चौड़ाई में सरलीकृत किया जाता है, इससे संबंधित अचानक परिवर्तन का मात्रात्मक रूप से विश्लेषण किया गया है। यह स्पष्ट हुआ है कि पल्स फोर्सिंग टर्म का आयाम जितना बड़ा होगा, एट्रैक्टर से प्रक्षेप पथ का विचलन उतना ही अधिक होगा और अचानक परिवर्तन उतना ही अधिक विनाशकारी होगा। इसके अलावा, पल्स फोर्सिंग टर्म की चौड़ाई जितनी अधिक होगी, उतनी ही लंबी अवधि जिसमें प्रक्षेप पथए ट्रैक्टर से भटक जाएगा। अंत में, दो सरल लेकिन सार्थक रैखिक संबंध का पता चला हैं: पहला पल्स फोर्सिंग टर्म के आयाम और एट्रैक्टर से प्रक्षेप पथ विचलन की दूरी के बीच, और दूसरा पल्स फोर्सिंग टर्म की चौड़ाई और उस अवधि के बीच जब प्रक्षेप पथ एट्रैक्टरसे बाहर रहता है। ये संबंध दर्शाते हैं कि अरैखीय प्रणाली में कुछ रैखिक गुण होते हैं।**

ABSTRACT. In this paper, preliminary theoretical research on abrupt change induced by the forcing term in a dynamical system is described. Taking the Lorenz equation trajectory as the research object, the trajectory response to different pulse forcing terms is studied based on the stability theorem of differential equations and numerical methods. From the perspective of a dynamical system, abrupt changecan be classified as internal or external. The former reflect strajectory self-adjustment inside the attractor, whereasthe latter represents the bizarre behavior of the trajectory in its deviation from the attractor. This classification helps in understanding the physical mechanisms of different manifestations of atmospheric abrupt change. For different intensities and durations of the pulse forcing term, which are simplified to the magnitude and width of a rectangular wave, respectively, the corresponding abrupt change is analyzed quantitatively. It is established that the larger the amplitude of the pulse forcing term, the greater the deviation of the trajectory from the attractor and the more violent the abrupt change. Moreover, the greater the width of the pulse forcing term, the longer the duration over which the trajectory deviates from the attractor. Finally, two simple but meaningful linear relationships are obtained : one between the amplitude of the pulse forcing term and the distance of trajectory deviation from the attractor and the otherbetween the width of the pulse forcing term and the duration over which the trajectory dwells outside of the attractor. These relationships indicate that nonlinear systems have some linear properties.

Key words – Abrupt change classification, Dynamical system, Forcing term, Stability analysis, Lorenz equation.

1. Introduction

Abrupt change is a phenomenon common both in nature and in human production activities, which can be manifested as either gradual or jumping change. The occurrence of earthquakes, biological variability and human mood swings are familiar examples of abrupt change.

Catastrophe theory, which is a part of chaos theory, is able to explain abrupt change as a jumping change that can cause qualitative change when using a mathematical model associated with chaos theory. The establishment of catastrophe theory can be traced back to 1953 when Hadamard solved the Cauchy problem of the Laplace equation. Having shown that the differential equation is highly sensitive to the initial value, which means the solution is unstable, he constructed a famous counter example (Daniell, 1953). In 1963, Lorenz proposed an equation that has clear physical meaning, which indicated that instability is not simply an abstract mathematical theory but that it also exists in nature and that the feature of such an unstable system is the uncertainty of trajectory motion (Lorenz, 1963). In 1974 French mathematician Thom undertook systematic research that laid the foundation of catastrophe theory (Thom, 1974). Subsequently, British mathematician Zeeman developed and improved the basic catastrophe theory (Zeeman, 1975). This theory has been used widely in many different subject areas such as genetic mutation in biology, adjustment of industrial structures in economics and climatic change in atmospheric science. All applications of this theory hope to exploit its capability of forecasting change in a complex and disordered system.

The interest of the atmospheric science community in abrupt change stems from the abrupt changes observed in the atmosphere that are jumping features of initial empirical abrupt change (Krishnamurti and Ramanathan, 1979; Lanzante, 1983; Shinoda, *et al*., 1986 and Mcbride, 1987). Climatic abrupt change and its associated theoretical research represent a new field of modern climatology that has received considerable attention from researchers around the word (Charney *et al*., 1979; Fu *et al*., 2003; Cavalcante *et al*., 2013) and for which various abrupt change types and detection methods have been proposed. Heuristic segmentation algorithm and approximate entropyare used to detect the abrupt changes of nonlinear time series (Feng *et al*., 2005; Wang and Zhang, 2008; Gong *et al*., 2017). The prominent features of atmospheric motion is nonlinearity, nonlinear method can be used to detect the abrupt climate change from the point of view of dynamic (Liu *et al*., 2015; Huang *et al*., 1993). For abrupt changes of yearly air temperature series in China, the Northern Hemisphere and the Globe are detected with a statistical test, It turns out that an abrupt change from a warm period to a cold one of the temperature in China occurred during the end of the 1940s and the beginning of 1950s and two abrupt changes of the temperature in the Northern Hemisphere and the Globe also happened both in the 1890s and the 1920s (Wei and Cao, 1995; Zheng *et al*., 2012).

Based on the station observation data from 1961 to 2006 in China, the trends and time points of abrupt change for surface air temperature (SAT) and precipitation are analyzed (Ding and Zhang, 2008). On behalf ofthe properties of operators of the equations, abrupt changes are also studied in the infinite dimensional Hilbert space (Li *et al*., 1996; Dingand, 2008).

In 1992, Fu, *et al*., proposed a universal definition of climatic abrupt change: “the abrupt change is the jumpy transformation phenomenon from one steady state (the stable and sustainable change trend) to another steady state (the stable and sustainable change trend), the performance is a sharp change from one statistical feature to another statistical feature in space and time” (Fu and Wang, 1992).

Atmospheric motion, which can be described by nonlinear fluid dynamics equations, reflects an unstable system that is sensitive to the initial field and parameters. As with the Lorenz equation, the trajectory can hop in a disorderly manner between two or more equilibrium states, which can lead to statistically sharp changes in time series of meteorological elements. Thus, climatic abrupt change can be considered a manifestation of the instability of the dynamics equations of the atmosphere, which is a reflection of the internal characteristics of atmosphere motion. Therefore, we can assess abrupt change based on the trajectory change of the dynamics equations of the atmosphere. In other words, if the trajectory moves within the area of one equilibrium state, there will be no abrupt change. Conversely, a trajectory jumping between different equilibrium states or exhibiting the behavior of

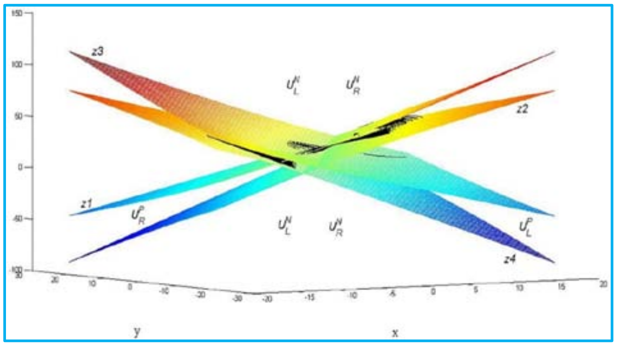


Fig. 1. Stable and unstable regions of the Lorenz equation [3]

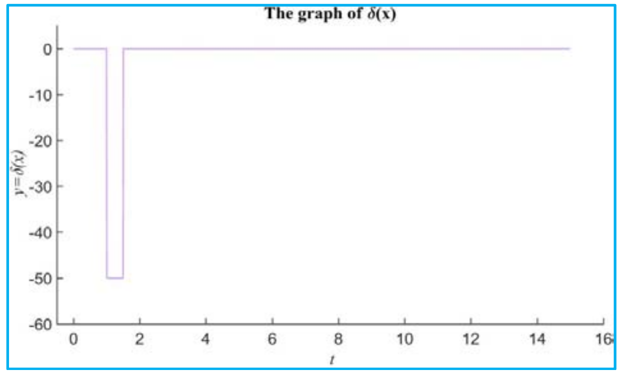


Fig. 2. Graph of pulse function δ(*t*)

moving away from the attractor can be regarded as the occurrence of abrupt change (Da *et al*., 2014; Shen *et al*., 2018). In this paper, taking the Lorenz equation as the research object, the trajectory response to different forcing terms is studied to elucidate a dynamics-based explanation for the mechanism of abrupt change, which could contribute to the study of climate change.

2. Theoretical foundation

The Lorenz system comprises the following set of nonlinear equations:

$$\begin{cases} \dfrac{dx}{dt} = 10(-x+y) \\ \dfrac{dy}{dt} = 28x - y - xz \\ \dfrac{dz}{dt} = xy - \dfrac{8}{3}z \end{cases} \tag{1}$$

It has two equilibrium points, *i.e*., $L\left(-6\sqrt{2},-6\sqrt{2},27\right)$ and $R\left(6\sqrt{2},6\sqrt{2},27\right)$, which are labeled left and right, respectively. Reference [3] describes the delineation of stable and unstable regions of the Lorenz equation. In Fig. 1, the region between surfaces *z*1 and *z*2, which is marked as U_R^P, is the unstable region of the right equilibrium points; theregion above *z*1 or below *z*2, which is marked as U_R^P, is the stable region of the right equilibrium points.Therefore, *z*1 and *z*2 representthe boundary surfaces of the stable and unstable regions of the right equilibrium points (and similarly for *z*3 and *z*4 for the left equilibrium points). In the stable region, the trajectorycannotmove away from the equilibrium point region; conversely, in the unstable region, the trajectory will move away from this equilibrium point to another (Da *et al*., 2014).

3. Numerical experiment

3.1. *Abrupt change induced by a disturbance forcing term*

A disturbance that is a pulse function $\delta(t)$, which has the form of a rectangular wave, can be expressed as follows:

$$\delta(t) = \begin{cases} -50 & 1 \le t \le 1.5 \\ 0 & \text{other} \end{cases} \tag{2}$$

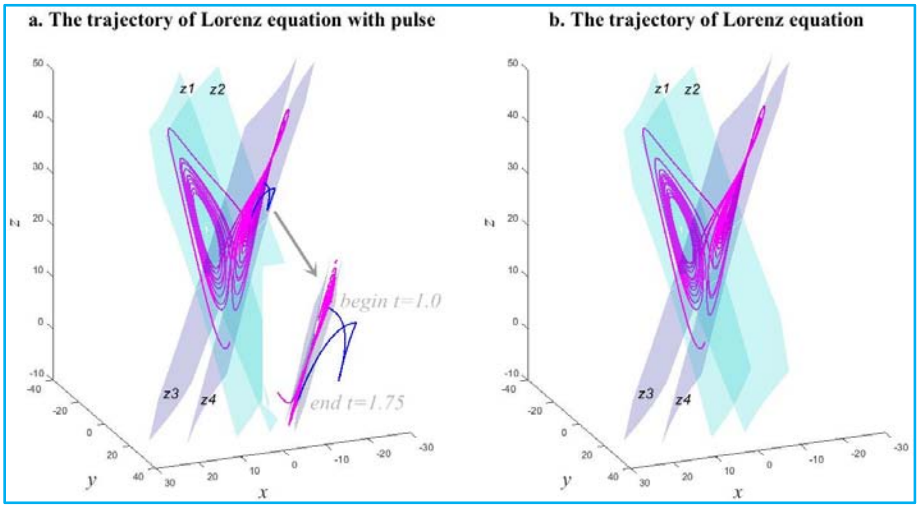
and depicted as in Fig. 2.

Taking the rectangular wave expressed in Eqn. (2) as a forcing term and adding it to the right-hand side of the first equation of the Lorenz equations [Eqn. (1)], we obtain a Lorenz system with apulse forcing term:

$$\begin{cases} \dfrac{dx}{dt} = 10(-x+y) + \delta(t) \\ \dfrac{dy}{dt} = 28x - y - xz \\ \dfrac{dz}{dt} = xy - \dfrac{8}{3}z \end{cases} \tag{3}$$

By taking the initial field as (1,2,3), which can be random,numerical solutions of dynamical systems (1) and (3) can be solved. Using the four-rank Runge-Kuttaalgorithms with an incremental step of 0.01 over the integral interval [0,20] and a truncation error of 0.01^3, surfaces *z*1 and *z*2 were found to have the following form ofanalytic expression (for further details, see Ref. [3]):

$$F_1(x, y, z) = -13.8546\,(0.8450x - 0.4380y - 0.3322z + 5.5159)^2 + 0.0940\,(-0.3313x + 0.1718y - 0.8529z + 24.3817)^2 + 0.0940\,(0.3650x + 1.0534y - 0.0880z - 9.6587)^2 \tag{4a}$$



Figs. 3(a&b). Trajectory of the Lorenz equation: (a) with pulse forcing and (b) without pulse forcing

Similarly, surfaces $z3$ and $z4$ have the following form of analytic expression:

$$F_2(x, y, z) = -13.8546\,(0.8450x - 0.4380y + 0.3322z - 5.5159)^2 + 0.0940\,(-0.3313x + 0.1718y + 0.8529z - 24.3817)^2 + 0.0940\,(0.3650x + 1.0534y + 0.0880z + 9.6587)^2 \quad (4b)$$

Based on the above, numerical experiments can produce the trajectories of the Lorenz equation presented in Fig. 3.

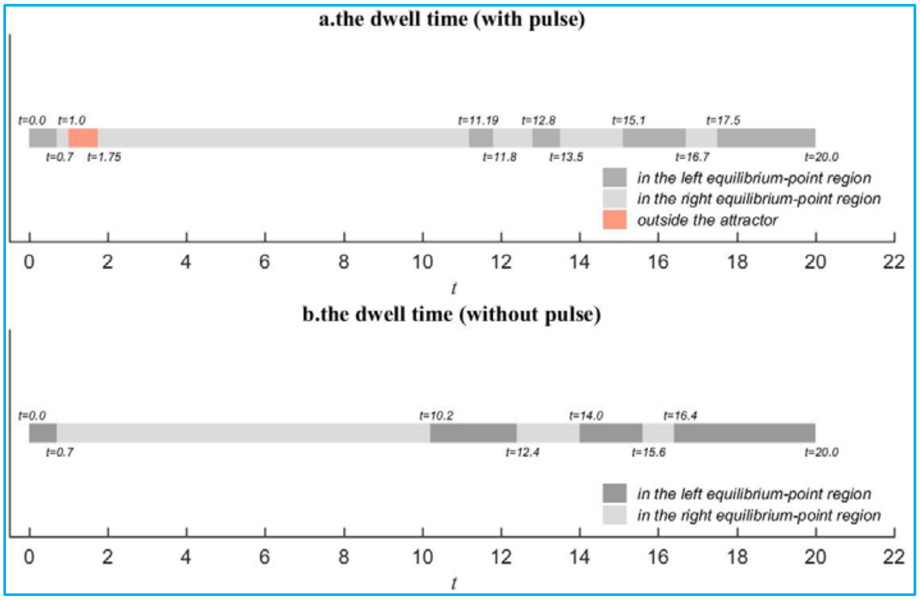
The trajectory of the Lorenz equation with pulse forcing, *i.e.*, Eqn. (3), is shown in Fig. 3(a) as a magenta line (with local highlighting in blue), together with surfaces $z1$ and $z2$ shown in translucent blue and surfaces $z3$ and $z4$ depicted in translucent cyan. In the lower-right corner, an enlarged localized graph is presented. Thetrajectory of the Lorenz equation without pulse forcing, *i.e.*, Eqn. (1), is shown in Fig. 3(b) using the same color scheme. A markedbulge (blue)is evident in the trajectory shown in Fig. 3(a), whereas there is no such anomaly in Fig. 3(b). The bulge shown in Fig. 3(a) starts at $t = 1.0$ and ends at $t = 1.75$, *i.e.*, the duration is 0.75 time units. Comparison of Eqns. (1) and (3), with reference to Figs. 3(a&b), reveals that the bulge is induced by the pulse function $\delta(t)$ and that the start and end times are also related to the pulse function but with a slight lag. Moreover, it can be seen that the trajectory density differs between Figs. 3(a&b).

In Ref. [3], an abrupt change is defined as the process of a dynamic system jump of the trajectory from one equilibrium state to another. This phenomenon is seen as the adjustment of the trajectory inside the attractor; the attractor of Lorenz system (1) is the “butterfly's wings” depicted in Fig. 3. As shown in Fig. 3(a), when the forcing term $[\delta(t)]$ is added to dynamic system (1) [*i.e.*, Eqn. (3)], the corresponding trajectory has a bulge. This characteristic is markedly different from the jumping between different equilibrium point regions; it reflects the movement of the trajectory away from the attractor, which has a different dynamic mechanism. Jumping between different equilibrium point regions represents self-adjustment of the trajectory within the attractor, whereas movement of the trajectory away from the attractor iscaused by the forcing term. To distinguish them, we call the former an internal abrupt change and the latter an external abrupt change. Based on the above, we can conclude that the pulse forcing term can lead to the occurrence of external abrupt change.

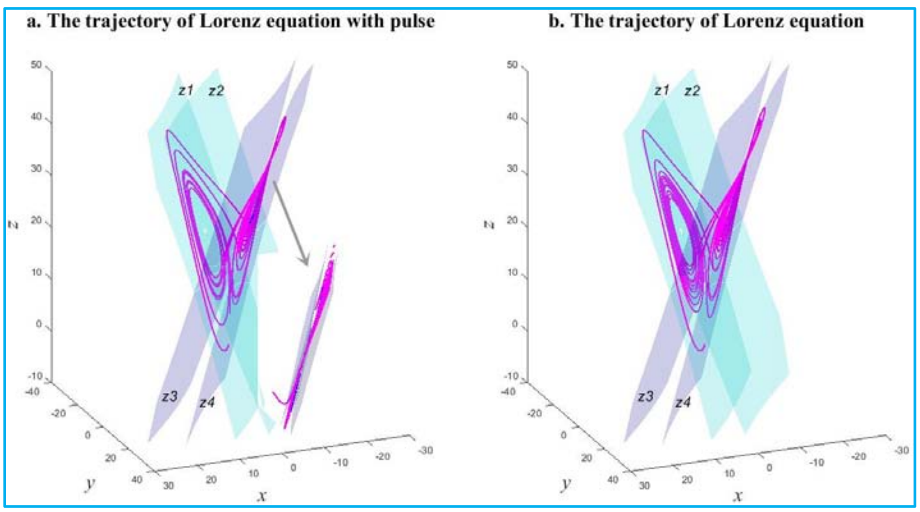
3.2. *Dwell time changeand pulse forcing term*

Comparison of the sparseness of the trajectory in the equilibrium point regions of Figs. 3(a&b) reveals the trajectory is sparse in the left equilibrium region of Fig. 3(a) but dense in the left equilibrium region. This means the pulse forcing term can also lead to change in the dwell time of the trajectory in the equilibrium point regions. For this reason, quantitative study on the dwell time of the trajectory in different equilibrium point regions is interesting.

Dwell time of the trajectory in Fig. 3(a) : time in the left equilibrium point region: [0, 0.7], [11.19, 11.8], [12.8,



Figs. 4(a&b). Dwell time of the trajectory (a) with pulse forcing and (b) without pulse forcing



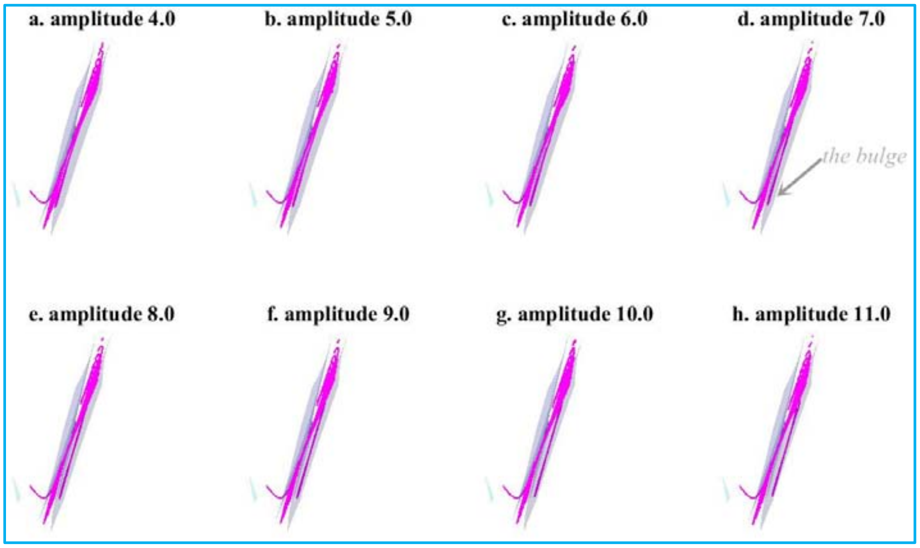
Figs. 5(a&b). Trajectory of the Lorenz equation: (a) with pulse forcing and (b) without pulse forcing

13.5], [15.1, 16.7], [17.5, 20.0]; time in the right equilibrium point region : [0.7, 1.0], [1.75, 11.19], [11.8, 12.8], [13.5, 15.1], [16.7, 17.5]; time outside the attractor [1.0, 1.75]. Dwell time of the trajectory in Fig. 3(b) : time in the left equilibrium point region: [0, 0.7], [10.2, 12.4], [14.0, 15.6], [16.4, 20.0]; time in the right equilibrium point region : [0.7, 10.2], [12.4, 14.0], [15.6, 16.4]. For an intuitive visualization, please see Figs. 4(a&b).

In Figs. 4(a&b), the length of the dark (light) gray rectangular block represents the dwell timeof the trajectory in the left (right) equilibrium point region (start and end times are marked on opposite sides) andthe orange rectangular block represents the time during which the trajectory is outside the attractor when the external abrupt change occurs. Comparison of the dwell times in the different equilibrium point regions reveals that the pulse forcing term not only induces external abrupt change but can also change the time at which internal abrupt change occurs.

3.3. *Response of external abrupt change to magnitude of pulse forcing term*

To explore the characteristics of the externalabrupt change, we consider the magnitude of the pulse forcing



Figs. 6(a-h). Trajectory of the Lorenz equation with pulse amplitude: (a) 4.0, (b), 5.0, (c) 6.0, (d), 7.0, (e) 8.0, (f) 9.0, (g) 10.0 and (h) 11.0

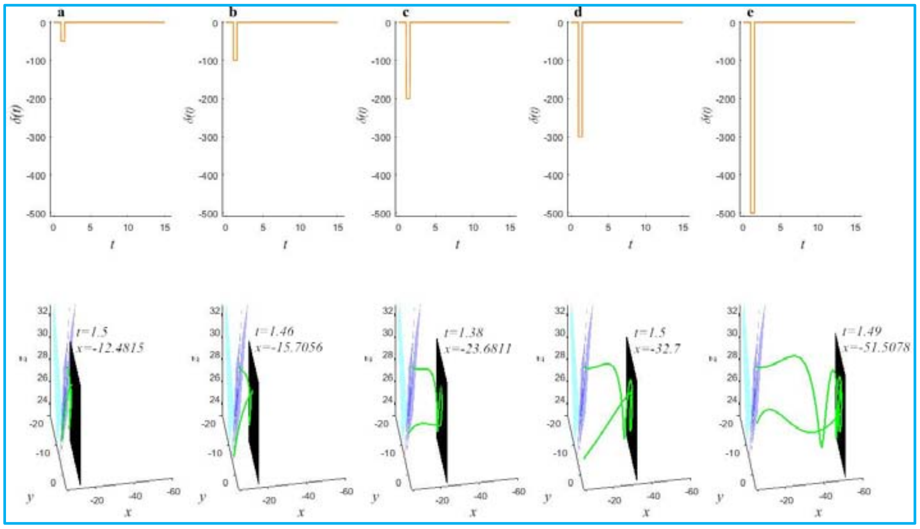
term to be 0.1. Thus, the pulse function $\delta(t)$ has the following form:

$$\delta(t) = \begin{cases} -0.1 & 1 \le t \le 1.5 \\ 0 & \text{other} \end{cases} \tag{5}$$

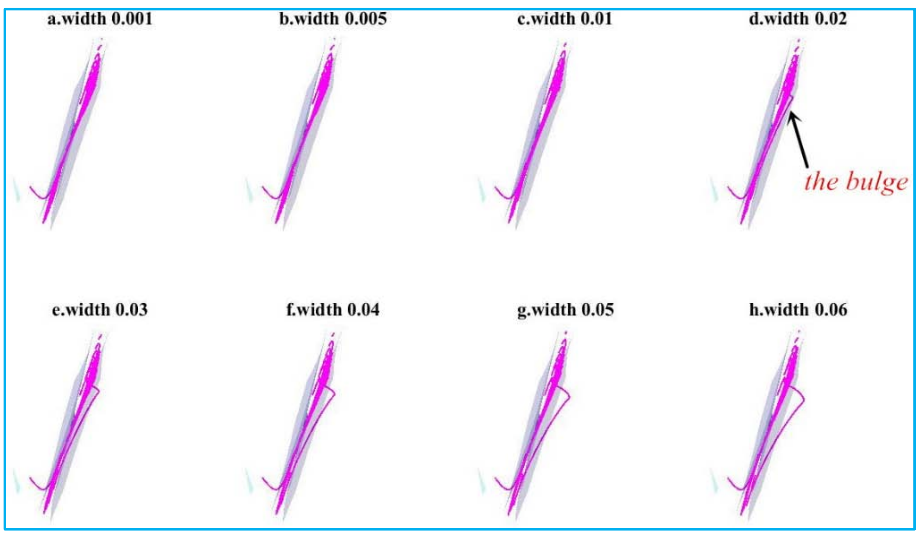
Using the same initial field and calculation method as described in the previous section, numerical solutions of dynamical systems (3) and (1) can be solved but by replacing pulse function (2) with Eqn. (5). The numerical solutions are shown in Fig. 5 using the same color scheme as in Fig. 3. It can be seen that the bulge evident in Fig. 3(a) does not appear prominently in Fig. 5(a). It means that pulse function (5) cannot induce the external abrupt change depicted in Fig. 3(a), *i.e.*, not all pulse functions will induce externalabrupt change. Comparison ofthe sparseness of the trajectory shown inside the attractor in Figs. 5(a&b) reveals that they are different. It indicates that although pulse function (5) cannot induce an external abrupt change, it can change the time at which internal abrupt changeoccurs.

The magnitudes of forcing terms (2) and (5) are different and therefore the occurrence of the external abrupt change might be related to the magnitude of the pulse. To investigate the threshold at which pulse amplitude can trigger external abrupt change, we analyzed the relationship between external abrupt change and pulse amplitude systematically. For the analysis, pulse amplitude was taken as 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0 and 11.0 and the corresponding local trajectory is shown in Figs. 6(a-h), respectively. In Figs. 6(a-c), the trajectory is inside the attractor, *i.e.*, there is no bulge on the trajectory. In Fig. 6(d), there is a small bulge but it is not obvious. In Fig. 6(e), the bulge is slightly more evident and it becomes increasingly pronounced at the pulse amplitude increases from 9.0 to 11.0 [Figs. 6(f-h), respectively]. Thus, as pulse amplitude increases, the bulge grows from nothing to something and from small to large. It means that any pulse term will inevitably induce abrupt change, either internal or external. If the pulse magnitude is small, internal abrupt change will be induced. Only when the pulse amplitude is sufficiently large will external abrupt change be triggered. It means that the occurrence of external abrupt change has a certain threshold in terms of pulse amplitude; only if the threshold is exceeded will external abrupt change occur.

The forcing term can be the force density function, which is related to momentum in differential equation theory. Of course, in the thermodynamic equation, it can also be the source or sink function. From this perspective, external abrupt change requires that the pulse term be sufficiently large, *i.e.*, the momentum must be sufficient to leverage internal system changes and induce external abrupt change. This could have implications for abrupt change within the atmospheric system. In the atmospheric system, only external atmospheric forcing is sufficiently large to promote external abrupt change. If the forcing is too small, external abrupt change cannot be induced; however, the time at which internal abrupt change occurs will be changed.

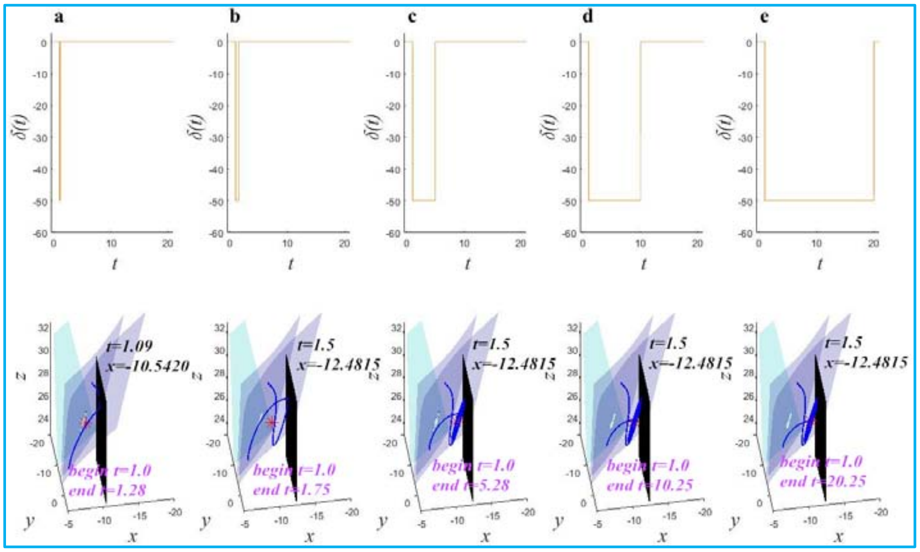


Figs. 7(a-e). Amplitude of pulse function δ(*t*) (upper row): (a) 50, (b) 100, (c) 200, (d) 300 and (e) 500 and local trajectory of the Lorenz equation (lower row)



Figs.8(a-h). Trajectory of the Lorenz equation with pulse width: (a) 0.001, (b) 0.005, (c) 0.01, (d) 0.02, (e) 0.03, (f) 0.04, (g) 0.05 and (h) 0.06

Numerical solutions of the dynamic systems with pulse amplitudes of 50.0, 100.0, 200.0, 300.0 and 500.0 [Eqn. (2)] are depicted in Fig. 7. In this figure, the panels in the upper (lower) row show the pulse function $\delta(t)$ (corresponding local trajectory). The lower panels show a green curve that represents the time during the interval [1, 1.75], local surfaces *z*1, *z*2, *z*3 and *z*4 and a black plane that represents $x = -12.4815$, *i.e.*, the furthest positionin the *x*-axis direction that the trajectory can reach when $t = 1.5$. In this instance, the black plane can partially characterize the extent of the external abrupt change. Obviously, the further the trajectory reaches, the more severe the external abrupt changewill be. When the pulse amplitude is 100 [Fig. 7(b)], the black plane is positioned at $x = -15.7056$ and $t = 1.46$. The position of the plane and the time in Figs. 7(c-e) are $x = -23.6811$, $t = 1.38$; $x = -32.7$, $t = 1.5$; and $x = -51.5078$, $t = 1.49$, respectively. It can be seen that the larger the pulse amplitude, the greater the deviation of the trajectory from the attractor and the more severe the external abrupt change. Irrespective of the magnitude of the deviation of the trajectory from the attractor, the time taken to move from the attractor to the furthest point is approximately 1.5 units and the time taken to return to the attractor is approximately 0.25 units. In terms of trajectory length, it can be seen that as pulse amplitude increases, the



Figs. 9(a-h). Width of pulse function (upper row): (a)0.1, (b) 0.4, (c) 4.0, (d) 9.0 and (e) 19.0 and local trajectory of the Lorenz equation (lower row)

trajectory length increases, which causes the speed to increase. It can also be seen that the duration of trajectory deviation from the attractor remains at approximately 1.75 time units, which means pulse amplitude affects the degree of external abrupt change but not the duration. Each of the trajectories shown in Figs. 7(c-e) presents a spiral structure when moving from the attractor to the furthest point and a quasilinear structure when returning.

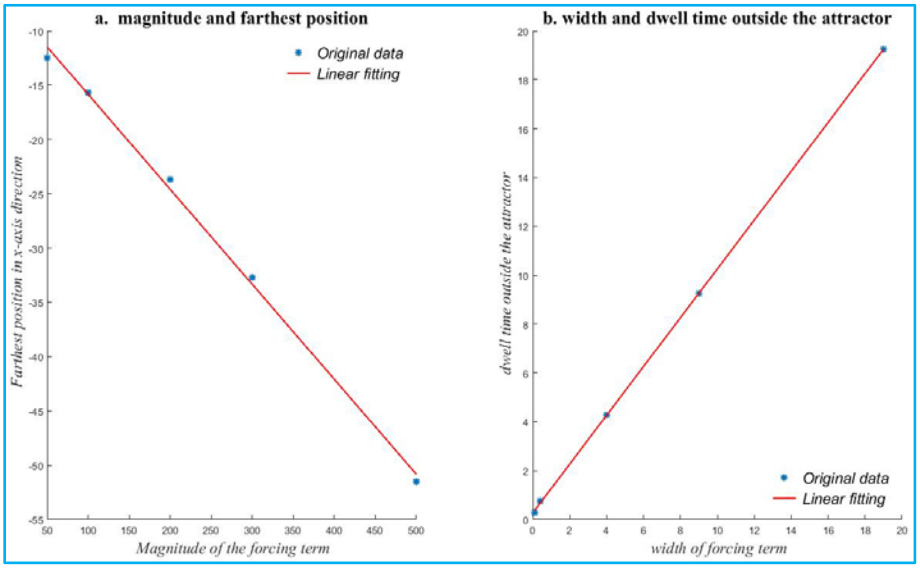
3.4. *Response of external abrupt change to width of pulse forcing term*

To analyze the impact of the width of the pulse forcing term on external abrupt change, pulse width was taken as 0.001, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05 and 0.06 and the corresponding local trajectory is shown in Figs. 8(a-h), similar to Fig. 6. A faint bulge can be seen when the pulse width is 0.02 [Fig. 8(d)] and the size of the bulge increases gradually with increasing pulse width [Figs. 8(e-h)]. Similar to Fig. 6, it can be seen that as the pulse width increases, the bulge grows from nothing to something and from small to large.

Taking the width of the forcing term as 0.1, 0.4, 4.0, 9.0 and 19.0, numerical solutions of the dynamical systems are illustrated in Fig. 9. The panels in the upper row of the figure depict the pulse function $\delta(t)$ and the lower panels present the local trajectory and the local surfaces $z1$, $z2$, $z3$ and $z4$, as in Fig. 7. In Fig. 9(a) (pulse width: 0.1), the local trajectory (blue line) jumps out of the attractor at t = 1.0, which is when the external abrupt change begins. At t = 1.28, the trajectory jumps back into the attractor and the external abrupt change ends; the dwell time outside the attractor is 0.28 time units. The black plane ($x = -10.5420$) is the furthest position that the trajectory can reach in the x-axis direction, which occurs when t = 1.09. For a pulse width of 0.4, the dwell time of the trajectory outside the attractor is [1, 1.75] and when t = 1.5, the furthest position that the trajectory can reach is the black plane at $x = -12.4815$ [Fig. 9(b)]. For the remaining examples of pulse width, *i.e*., 4.0, 9.0 and 19.0, the time interval is [1.0, 5.28], [1.0, 10.25] and [1.0, 20.25]; the black plane is at $x = -12.4815$, $x = -12.4815$ and $x = -12.4815$; the time t = 1.5, t = 1.5 and t = 1.5; and the dwell time outside the attractor is 4.28, 9.25 and 19.25, respectively [Figs. 9(c-e)]. It can be seen that the larger the pulse width, the longer the trajectory deviates from the attractor and the longer the external abrupt change lasts; however, the furthest position that the trajectory can reach remains largely unchanged. Therefore, the dwell time outside the attractor is determined by the pulse width, whereas the furthest position that the trajectory can reach is controlled by the pulse amplitude.

3.5. *Linear relationship in nonlinear systems*

Taking the magnitude of the forcing term as the x-axis and the furthest position that the trajectory reaches as the y-axis, the data from Fig. 7 [*i.e*., (50, −12.4815), (100, −15.7056), (200, −23.6811), (300, −32.7) and (500, −51.5078)] are plotted as blue stars in Fig. 10(a). The red fitting line suggests a strong linear relationship between



Figs. 10(a&b). Linear relationship in nonlinear systems: (a) pulse magnitude and furthest position and (b) pulse width and dwell time outside the attractor

the pulse amplitude and the furthest position that the trajectory can reach; such a linear relationship in a nonlinear system is rare.

Taking the width of the forcing term as the *x*-axis and the dwell time of the trajectory outside the attractor as the *y*-axis, the data from Fig. 9 [*i.e.*, (0.1, 0.28), (0.4, 0.75), (4, 4.28), (9, 9.28) and (19, 19.25)] are plotted as blue stars in Fig. 10(b). Again, the red fitting line indicates astrong linear relationship between the pulse width and the dwell time of the trajectory outside the attractor.

4. Conclusions and Outlook

Here, we investigated the response of the Lorenz equation trajectoryto an idealized forcing termcomprising a single pulse, in which the pulse amplitude represented intensity and the pulse width represented duration. For different pulse forcing terms, the response of the trajectory was discussed and the following conclusions derived.

(*i*) Abrupt change was classified as either internal or external. The former represents the trajectory jumping between different equilibrium point regions, while the latter reflects the movement of the trajectory away from the attractor.

(*ii*) Classically, the forcing term can induce abrupt change; however, only if the amplitude or width exceeds a certain threshold can external abrupt change occur. This feature should not be limited solely to the Lorenz equation but true also for a general dynamical system.

(*iii*) The larger the pulse amplitude, the further the trajectory deviates from the attractor and the more severe the external abrupt change. The larger the pulse width, the longer the duration of the deviation of the trajectory from the attractor and the longer the external abrupt change persists. No significant relationship was found between the degree of the external abrupt change and the width of the forcing term, or between its duration and the amplitude of the forcing term.

(*iv*) It was found that certain linear properties exist in the nonlinear system investigated.

Must to say, as the numerical solution of differential equations can be smooth by adjusting integration step size, our research is confined on theoretical aspects which is based on numerical solutions of differential equations. For the study on climate change, we have made some attempts, but the conclusion obtained is not very ideal,the reason is that the meteorological data are not smooth enough. For the use in climate change research, it is necessary to smooth meteorological data.

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Author Contributions

ChaoJiu Da wrote the main text of the manuscript and undertook most of the theoretical research. Tai Chen Feng wrote the introduction of the manuscript and checked the English grammar in the remainder of the article. BingLu Shen designed and implemented all numerical experiments. Jian Song contributed to the scientific discussion. All authors reviewed the manuscript.

Competing Interests

For both financial and non-financial interests, the authors declare no competing interests.

Data Availability Statement

The data that supports the findings of this study are available within the article.

Disclaimer : The contents and views expressed in this study are the views of the authors and do not necessarily reflect the views of the organizations they belong to.

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