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Analytical treatment for the fractional advection diffusion equation in three dimensions

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सार – **यह कार्य वायुमंडलीय परिसीमा स्तर (एबीएल) में दूषित पदार्थों के विसरण का अनुकरण करने के लिए त्रि-आयामी अभिवहन-विसरण समीकरण की विश्लेषणात्मक अभिक्रिया प्रस्तुत करता है न कि भिन्नात्मक क्रम** α **(गैर-पूर्णांक क्रम) के पहले के रूप** (x, y, z) **में दो आयामों में, जैसा पहले होता था। इस अभिक्रिया में, अनुदैर्ध्य पवन की गति** u **और ऊर्ध्वाधर भंवर विसरणशीलता** K_z **को केवल ऊर्ध्वाधर ऊंचाई** z **पर निर्भर माना जाता है, जबकि तिर्यक पवन भंवर विसरणशीलता** K_y **को स्रोत से नीचे की दूरी** x **और जमीन के ऊपर ऊर्ध्वाधर ऊंचाई** z **पर निर्भर माना जाता है। जेनरलइज्ड इंटीग्रल लैपलेस ट्रांसफॉर्म तकनीक (जीआईएलटीटी) का उपयोग करके विश्लेषणात्मक समाधान प्राप्त किया गया है जो अभी भी कैपुटो के भिन्नात्मक व्युत्पन्न सूत्र के उपयोग पर आधारित एक अच्छी विधि है।** α **के भिन्न-भिन्न भिन्नात्मक मूल्यों के लिए प्रस्तावित मॉडल और कोपेनहेगन प्रयोग सांद्रता के बीच तुलना का मूल्यांकन किया गया। प्रस्तावित और प्रयोगात्मक सांद्रता के बीच सांख्यिकीय विश्लेषण से पता चलता है कि सर्वोत्तम परिणाम भिन्नात्मक क्रम** $\alpha = 0.95$ **के साथ प्राप्त किए गए, इस अध्ययन के परिणामों पर चर्चा की गई है और तालिकाओं और चित्रों द्वरा प्रस्तुत किया गया है।**

ABSTRACT. This work presents an analytical treatment of the three-dimensional advection-diffusion equation in (x, y, z), not two dimensions as before, of fractional order α (non-integer order) to simulate the dispersion of contaminants in the atmospheric boundary layer (ABL). In this treatment, the longitudinal wind speed u and the vertical eddy diffusivity K_z are taken to be dependent only on the vertical height z, while the crosswind eddy diffusivity K_y is taken to be dependent on the downwind distance x from the source and the vertical height z above ground. The analytical solution has been derived using the Generalized Integral Laplace Transform Technique (GILTT) which is still a good method, adopting Caputo's formula of a fractional derivative. The comparison between the proposed model for different fractional values of α and the Copenhagen experiment concentrations was evaluated. The statistical analysis between the proposed and experimental concentrations reveals that the best results were obtained with fractional order $\alpha = 0.95$ than others, the results of this study are discussed and presented in tables and illustrative figures.

Key words – Generalized Integral Laplace Transform Technique (GILTT), Caputo's formula, Copenhagen experiment, Atmospheric Boundary Layer (ABL), Fractional order.

1. Introduction

The dispersion of pollutants in the atmosphere is governed by the atmospheric advection-diffusion equation. The dispersion models based on the analytical solutions of the advection-diffusion equation (ADE) are of fundamental importance in understanding and describing physical phenomena. The analytical model has many advantages over the numerical solution, since all parameters appear explicitly in the solution, so that their effect can be easily investigated. Also, it is useful in examining the accuracy of numerical model. Several efforts have been made to find the analytical solution of the ADE with traditional derivatives (integer order) using different methodologies (Yeh and Huang, 1975; Sharan and Yadav, 1998; Moreira *et al*., 2005, 2009, 2014; Wortmann *et al*., 2005; Tirabassi *et al*., 2008; Essa *et al*., 2007, 2014, 2016; Sharan and Kumar, 2009; Buske *et al*., 2012; Marie *et al*., 2015; Sharan *et al*., 1996).

Recently, special attention has been devoted to find the analytical solution of the advection-diffusion equation of fractional order (non-integer order) to simulate the pollutant dispersion in the atmospheric boundary layer (Xavier *et al*., 2019; Moreira and Moret, 2018; Goulart *et al*., 2017; Matlob and Jamali, 2019; Rubbab *et al*.,

2016). Numerical analysis of the fractional evolution model for heat flow in materials with memory has been studied by Nikan *et al*. (2020), A computational approach for the space-time fractional advection-diffusion equation arising in contaminant transport through porous media has been evaluated by Esmaeelzade Aghdam *et al*. (2020), Also, Numerical approach for modeling fractal mobile/immobile transport model in porous and fractured media has been studied by Nikan *et al*. (2020), Numerical approximation of the time fractional cable model arising in neuronal dynamics has been investigating by Nikan *et al*. (2020), Numerical evaluation of fractional Tricomi-type model arising from physical problems of gas dynamics has been solved by Nikan *et al*. (2020).

In this work we present an analytical investigation for the three-dimensional fractional advection-diffusion equation by using the Generalized Integral Laplace Transform Technique (GILTT), adopting Caputo's formula of a fractional derivative. Assuming in this treatment that the wind speed u and vertical eddy diffusivity K_z are functions only of the vertical height *z*, while the lateral eddy diffusivity K_y is taken to be dependent on the downwind distance *x* from the source and the vertical height *z* above ground. The proposed analytical formula has been compared with Copenhagen experiment concentrations data set. Statistical measures have been utilized in the comparison between the observed and proposed concentrations by the new model with different values of the fractional order α of the derivative. This study reveals that the concentrations obtained by the proposed model with α = 0.95 are in a very good agreement with those measured than others values. The results of this study are discussed and presented in tables and illustrative figures.

2. Solution of the fractional advection-diffusion equation

The steady state fractional advection-diffusion equation that describes the dispersion of a non-reactive pollutant released from a point source in a turbulent atmospheric boundary layer can be written as:

$$u\frac{{}^{C}\partial^{\alpha}\overline{C}}{\partial x^{\alpha}}+v\frac{\partial\overline{C}}{\partial y}+w\frac{\partial\overline{C}}{\partial z}=\frac{\partial}{\partial x}\left(K_x\frac{\partial\overline{C}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial\overline{C}}{\partial y}\right)\frac{\partial}{\partial z}\left(K_z\frac{\partial\overline{C}}{\partial z}\right),\quad 0<\alpha\le 1 \tag{1}$$

where, $\overline{C}$ (*x*, *y*, *z*) is the mean contaminant concentration (*Bq*/m^3) or (g/m^3), *c* is the Caputo derivative, α is the order of the fractional spatial operator, *u*, *v*, *w* and K_x, K_y, K_z are the components of wind speed (m/s) and eddy diffusivity coefficient (m^2/s) along the *x*, *y* and *z* directions, respectively.

On using the following assumptions: (1) The mean wind blowing along the *x* -axis, so that *v* and *w* = 0 and (2) The transport of pollutants due to diffusion in the *x* direction is neglected in compared to that due to advection, equation (1) is reduced to the following equation:

$$u\frac{{}^{C}\partial^{\alpha}\overline{C}}{\partial x^{\alpha}}=\frac{\partial}{\partial y}\left(K_y\frac{\partial\overline{C}}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial\overline{C}}{\partial z}\right) \tag{2}$$

Equation (2) can be solved under the following boundary conditions:

Zero flux in *z* direction at vertical height $z = z_o$ (roughness length) and ABL top.

$$K_z\frac{\partial\overline{C}}{\partial z}=0 \text{ at } z=z_o,\quad z=h \tag{3a}$$

Zero flux in *y* direction at crosswind height $y = y_o$ (height in *y* direction) and at L_y (large distance in the crosswind direction).

$$K_y\frac{\partial\overline{C}}{\partial y}=0 \text{ at } y=y_o,\quad y=L_y \tag{3b}$$

A source with emission rate Q at height, H_s :

$$u\overline{C}(x,y,z)=Q\delta(y-y_o)\delta(z-H_s) \text{ at } x=0 \tag{3c}$$

Considering that *u*, K_z are functions of *z* only and K_y is function of (*x*, *z*), where *h* is the height of the atmospheric boundary layer (*m*), z_o is the roughness length (*m*), L_y is a large distance in the crosswind direction (*m*), *Q* is the rate of emission (g/s) or (Bq/s), H_s. is the source height (*m*), δ(.) is the Dirac Delta function. The source position is at *x* = 0, $y = y_o$ and $z = H_s$.

Let us expand $\overline{C}(x,y,z)$ as a linear combination of orthogonal eigen functions $\psi_\ell(y)$; namely:

$$\overline{C}(x,y,z)=\sum_{\ell=0}^{M}\overline{C}_\ell(x,z)\frac{\psi_\ell(y)}{N_\ell^{1/2}} \tag{4}$$

In which $\psi_\ell(y)$ satisfy the differential equation:

$$\frac{d^2\psi_\ell(y)}{dy^2}+\lambda_\ell^2\psi_\ell(y)=0,\quad 0<y<L_y \tag{5}$$

and has the form:

$$\psi_\ell(y) = \cos(\lambda_\ell y) \tag{6}$$

where, $\psi_\ell(y)$ and λ_ℓ, are respectively the eigen functions and eigen values satisfy the following orthonormality relation:

$$\frac{1}{N_m^{1/2} N_n^{1/2}} \int_0^{L_y} \psi_m(y)\psi_n(y)dy = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if } m = n \end{cases} \tag{7}$$

where,

$$N_m = \int_0^{L_y} \psi_m^2(y)dy \tag{8}$$

Inserting Eqn. (4) in Eqn. (2), multiplying with $\psi_m(y)/N_m^{1/2}$ and integrating from 0 to L_y, with respect to y, yields:

$$\begin{aligned} &\sum_{\ell=0}^{M} \int_0^{L_y} \frac{\psi_\ell(y)\Psi_m(y)}{\sqrt{N_\ell N_m}} dy \; u(z) \frac{c\partial^\alpha \overline{C}_\ell(x,z)}{\partial x^\alpha} \\ &= \sum_{\ell=0}^{M} K_y(x,z)\overline{C}_\ell(x,z) \int_0^{L_y} \frac{\psi_m(y)}{N_m^{1/2}} \frac{1}{N_\ell^{1/2}} \frac{d^2\psi_\ell(y)}{dy^2} dy \\ &+ \sum_{\ell=0}^{M} \int_0^{L_y} \frac{\Psi_m(y)}{N_m^{1/2}} \frac{\Psi_\ell(y)}{N_\ell^{1/2}} \frac{\partial}{\partial z}\left[K_z(z) \frac{\partial}{\partial z} \overline{C}_\ell(x,z) \right] dy \end{aligned} \tag{9}$$

Upon using Eqn. (5), the above equation can be simplified to:

$$\begin{aligned} &\sum_{\ell=0}^{M} \alpha_{\ell m} \; u(z) \frac{c\partial^\alpha \overline{C}_\ell(x,z)}{\partial x^\alpha} \\ &= \sum_{\ell=0}^{M} -\lambda_\ell^2 \alpha_{\ell m} K_y(x,z)\overline{C}_\ell(x,z) \\ &+ \sum_{\ell=0}^{M} \alpha_{\ell m} \frac{\partial}{\partial z}\left[K_z(z) \frac{\partial \overline{C}_\ell(x,z)}{\partial z} \right] dy \end{aligned} \tag{10}$$

And can be written in matrix form as:

$$\begin{aligned} &u(z) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \end{pmatrix} \begin{bmatrix} \frac{c\partial^\alpha \overline{C}_0(x,z)}{\partial x^\alpha} \\ \vdots \\ \vdots \\ \frac{c\partial^\alpha \overline{C}_M(x,z)}{\partial x^\alpha} \end{bmatrix} \\ &+ \lambda_\ell^2 K_y(x,z) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{bmatrix} \overline{C}_0(x,z) \\ \vdots \\ \vdots \\ \overline{C}_M(x,z) \end{bmatrix} \\ &- \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{Bmatrix} \frac{\partial}{\partial z}\left[K_z(z) \frac{\partial \overline{C}_0(x,z)}{\partial z} \right] \\ \vdots \\ \vdots \\ \frac{\partial}{\partial z}\left[K_z(z) \right] \frac{\partial \overline{C}_M(x,z)}{\partial z} \end{Bmatrix} = 0 \end{aligned} \tag{11}$$

This yields a set of M + 1 two dimensional differential equations, namely:

$$\begin{aligned} u(z) \frac{c\partial^\alpha \overline{C}_\ell(x,z)}{\partial x^\alpha} = \frac{\partial}{\partial z}\left[K_z(z) \frac{\partial \overline{C}_\ell(x,z)}{\partial z} \right] - \\ \lambda_\ell^2 K_y(x,z) \overline{C}_\ell(x,z) \end{aligned} \tag{12}$$

In which $\alpha_{\ell m}$ stands for:

$$\alpha_{\ell m} = \int_0^{Ly} \frac{\psi_\ell(y)}{N_\ell^{1/2}} \frac{\psi_m(y)}{N_m^{1/2}} dy \tag{13}$$

and $\alpha_{\ell m}$, λ_ℓ by virtue of Eqns. (5, 6, 7 and 8) are given by:

$$\alpha_{\ell m} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases} \tag{14}$$

and

$$\lambda_\ell = \frac{\ell\pi}{L_y}, \qquad \ell = 0,1... \tag{15}$$

Also, $\overline{C}_\ell(x,z)$ can be expressed as a linear combination of orthogonal eigen functions $\varphi_m(z)$ as :

$$\overline{C}_\ell(x,z)=\sum_{m=0}^{M}\ \overline{C}_{\ell m}(x)\frac{\varphi_m(z)}{N_m^{1/2}} \tag{16}$$

where,

$$\varphi_m(z)=\cos(\mu_m Z),\ \ \mu_m=\frac{m\pi}{h},\ \ m=0,1,2,... \tag{17}$$

Substituting Eqn. (16) into Eqn. (12), multiplying with $\varphi_s(z)/N_s^{1/2}$ and integrating from 0 to h with respect to z yields:

$$\begin{aligned}&\sum_{m=0}^{M}\int_0^h u(z)\frac{\varphi_m(z)\varphi_s(z)}{\sqrt{N_m N_s}}dz\frac{{}^c\partial^\alpha\overline{C}_{\ell m}(x)}{\partial x^\alpha}\\&\sum_{m=0}^{M}\overline{C}_{\ell m}(x)\int_0^h\frac{\varphi_s(z)}{\sqrt{N_s N_m}}\frac{\partial}{\partial z}\left[K_z(z)\frac{\partial\varphi_m(z)}{\partial z}\right]dz\\&-\lambda_\ell^2\sum_{m=0}^{M}\ \overline{C}_{\ell m}(x)\int_0^h K_y(x,z)\frac{\varphi_s(z)\varphi_m(z)}{\sqrt{N_s N_m}}dz\end{aligned} \tag{18}$$

This can be simplified to:

$$\sum_{m=0}^{M}\ (B_1)_{ms}\frac{c d^\alpha\overline{C}_{\ell m}(x)}{dx^\alpha}+\sum_{m=0}^{M}\ (b_2)_{ms}\overline{C}_{\ell m}(x)=0 \tag{19}$$

In which $(B_1)_{ms}$, $(B_2)_{ms}$ and $\overline{C}_{\ell m}(x)$ represent the matrices B_1, B_2 and the column matrix $Y_\ell(x)$ namely :

$$(B_1)_{ms}=\int_0^h u(z)\frac{\varphi_m(z)\varphi_s(z)}{\sqrt{N_m N_s}}dz \tag{20}$$

$$\begin{aligned}(B_2)_{ms}=&\lambda_\ell^2\int_0^h K_y(x,z)\frac{\varphi_s(z)\varphi_m(z)}{\sqrt{N_s N_m}}dz\\&-\int_0^h\frac{\varphi_s(z)}{\sqrt{N_s N_m}}\frac{d}{dz}\left[K_z(z)\frac{d\varphi_m(z)}{dz}\right]dz\end{aligned} \tag{21}$$

$$Y_\ell(x)=\begin{bmatrix}C_{\ell o}(x)\\ \vdots\\ \vdots\\ C_{\ell M}(x)\end{bmatrix} \tag{22}$$

Then Eqn. (19) in matrix form reads:

$$B_1\frac{{}^c d^\alpha Y_\ell(x)}{dx^\alpha}+B_2 Y_\ell(x)=0\quad\text{ for }\ 0<\alpha\le 1,\ x>0 \tag{23}$$

and can be written as:

$$\frac{{}^c d^\alpha Y_\ell(x)}{dx^\alpha}+F\,Y_\ell(x)=0 \tag{24}$$

where, F is the matrix;

$$F=\mathrm{B}_1^{-1}\mathrm{B}_2 \tag{25}$$

B_1, B_2 are constant matrices.

The transformed problem given by equation (24) can be solved analytically by using the Laplace transform technique and diagonalization of the matrix F (Segatto and Vilhena, 1999; Moreira *et al*. 2009). Laplace transform technique transforms the variable x to s and the function $Y_\ell(x)$ to $\tilde{Y}_\ell(s)$, namely:

$$\tilde{Y}_\ell(s)=\int_0^\infty e^{-xs}Y_\ell(x)dx \tag{26}$$

and adopting the Laplace transform of a fractional derivative of α order given by Caputo's formula (Moreira and Moret, 2018):

$$\begin{aligned}L\left[D_x^\alpha f(x)\right]=&\ s^\alpha\tilde{F}(s)\\&-\sum_{k=0}^{n-1}\ s^{\alpha-k-1}f^{(k)}(0),\quad n-1<\alpha\le n\end{aligned} \tag{27}$$

Therefore, in this study we used:

$$L\left[D_x^\alpha f(x)\right]=s^\alpha\tilde{F}(s)-s^{\alpha-1}f(0),\qquad 0<\alpha\le 1 \tag{28}$$

Then, Eqn. (24) transformed to:

$$s^\alpha\tilde{Y}_\ell(s)+F\tilde{Y}_\ell(s)=s^{\alpha-1}Y_\ell(0) \tag{29}$$

To find the expressed expression of $\tilde{Y}_\ell(s)$, the matrix F in Eqn. (29) can be diagonalized as:

$$F=XDX^{-1} \tag{30}$$

where, D is the diagonal matrix of eigen values of the matrix F, X is the matrix of the respective eigen functions and X^{-1} is its inverse.

Therefore, Eqn. (29) becomes as:

$$\left(s^\alpha I+XDX^{-1}\right)\tilde{Y}_\ell(s)=s^{\alpha-1}Y_\ell(0) \tag{31}$$

After algebraic treatment we obtain:

$$\tilde{Y}_\ell(s) = X\left[\frac{s^{\alpha-1}}{s^\alpha I + D}\right] X^{-1} Y_\ell(0) \tag{32}$$

where, "*I*" is the identity matrix.

Performing the Laplace transform inversion on Eqn. (32) yields:

$$Y_\ell(x) = X\ G(x)\ X^{-1}\ Y_\ell(0) \tag{33}$$

$$\text{where, } G(s) = L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha I + D}\right] \tag{34}$$

It is a diagonal matrix and L^{-1} is the inverse Laplace transform.

Adopting the standard theory for the Laplace transformation yields (Matloband Jamali, 2019; Moreira and Moret, 2018):

$$G(x) = \begin{bmatrix} E_\alpha\left(-d_1 x^\alpha\right) & 0 & \cdots & 0 \\ 0 & E_\alpha\left(-d_2 x^\alpha\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_\alpha\left(-d_N x^\alpha\right) \end{bmatrix} \tag{35}$$

where, E_α is the Mittag-Leffler function, which is intrinsic to the solution of equations with fractional derivatives and has the form:

$$E_\alpha\left(-d_i x^\alpha\right) = \sum_{k=0}^{\infty} \frac{\left(-d_i x^\alpha\right)^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0 \tag{36}$$

where, d_i are the eigen-values of the matrix F and Γ is the Gamma function.

Therefore, the final form of Eqn. (32) can be rewritten as:

$$Y_\ell(x) = X\begin{bmatrix} E_\alpha\left(-d_1 x^\alpha\right) & 0 & \cdots & 0 \\ 0 & E_\alpha\left(-d_2 x^\alpha\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_\alpha\left(-d_N x^\alpha\right) \end{bmatrix} X^{-1} Y_\ell(0) \tag{37}$$

The column matrix $Y_\ell(0)$ can be evaluated by inserting Eqn. (4) in the boundary condition given by Eqn. (3c), multiplying with $\psi_i(y)/N_i^{1/2}$ and integrating with respect to *y* from 0 to L_y, gives:

$$u(z)\overline{C}_i(0,z) = Q\delta(z - H_s)\frac{\psi_i(y_o)}{N_i^{1/2}} \tag{38}$$

Then, introducing Eqn. (16) in Eqn. (38), multiplying with $\varphi_\ell(z)/N_\ell^{1/2}$ and integrating with respect to *z* from 0 to *h* to get:

$$\sum_{m=0}^{M} \overline{C}_{\ell m}(0)\int_o^h u(z)\frac{\varphi_\ell(z)\varphi_m(z)}{\sqrt{N_\ell N_m}}dz = \int_o^h Q\frac{\psi_\ell(y_0)}{\sqrt{N_\ell}}\delta(z - H_s)\frac{\varphi_\ell(z)}{\sqrt{N_\ell}}dz = Q\frac{\psi_\ell(y_0)\varphi_\ell(H_s)}{N_\ell} \tag{39}$$

and can be written in matrix form as:

$$B_1 Y_\ell(0) = Q\frac{\psi_\ell(y_0)\varphi_\ell(H_s)}{N_\ell} \tag{40}$$

Therefore, the column matrix $Y_\ell(0)$ can be written as:

$$Y_\ell(0) = \begin{bmatrix} \overline{C}_{\ell o}(0) \\ \vdots \\ \vdots \\ \overline{C}_{\ell M}(0) \end{bmatrix} = B_1^{-1} Q\frac{Q\psi_\ell(y_0)\varphi_\ell(H_s)}{N_\ell} \tag{41}$$

Or

$$Y_\ell(0) = \frac{Q\psi_\ell(y_0)\varphi_\ell(H_s)}{\int_o^h u(z)\varphi_\ell(z)\varphi_m(z)dz}\sqrt{\frac{N_m}{N_\ell}}, \quad \ell = 0,1,2,... \tag{42}$$

$Y_\ell(x)$ is a column matrix whose components are $\overline{C}_{\ell M}(x)$ and $Y_\ell(0)$ is a column matrix whose components are $\overline{C}_{\ell M}(0)$, therefore the two dimensional concentration given by Eqn. (16) is determined once $\overline{C}_{\ell M}(x)$ is known. Consequently, the three-dimensional concentration given by Eqn. (4) is determined once $\overline{C}_{\ell M}(x,z)$ is known.

TABLE 1

Meteorological parameters during the Copenhagen experiment

Exp.	$\overline{u}_{10}(ms^{-1})$	$\overline{u}_{115}(ms^{-1})$	$u_*(ms^{-1})$	$w_*(ms^{-1})$	- L(m)	h(m)
1	2.1	3.4	0.36	1.8	37	1980
2	4.9	10.6	0.73	1.8	292	1920
3	2.4	5.0	0.38	1.3	71	1120
4	2.5	4.6	0.38	0.7	133	390
5	3.1	6.7	0.45	0.7	444	820
6	7.2	13.2	1.05	2.0	432	1300
7	4.1	7.6	0.64	2.2	104	1850
8	4.2	9.4	0.69	2.2	56	810
9	5.1	10.5	0.75	1.9	289	2090

In the present study a power-law profile is used to describe the vertical variation of wind speed with height z above ground surface in the ABL and has the form (Essa and Maha, 2008):

$$u(z) = a\, z^{p}, \qquad z \neq 0 \tag{43}$$

where, a = 3 and the values of p are taken from Hanna *et al*., (1982).

The lateral eddy diffusivity K_y is taken to be dependent on the down winds source distance x and the vertical height z above ground surface and is given by (Essa and Maha, 2008):

$$Ky(x, z) = \beta\, ux / a, \qquad \beta = 0.31\,(w_* / u)^2 \tag{44}$$

While the vertical variation of the vertical eddy diffusivity K_z with height z above ground surface is defined as (Pleim and chang, 1992):

$$K_z(z) = kw_*z\left(1-\frac{z}{h}\right) \tag{45}$$

where, w_* is given by the following expression (Degrazia*et al*. 2001):

$$w_* = u_*\left(-\frac{h}{kL}\right)^{1/3} \tag{46}$$

where, w_* is the convective velocity, k = 0.4 is the von Karman constant, h is the height of the planetary boundary layer, u_* is the friction velocity and L is the Monin-Obukhov length scale. The power-law exponent p of wind speed is a function of the atmospheric stability and the nature of underlying surface.

2.1. *Evaluation of the model against Copenhagen data set*

The performance of the proposed solution has been evaluated against the observed data set from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark, under neutral and unstable conditions (Gryning and Lyck, 1984; Gryning *et al*., 1987). In Copenhagen experiment the tracer SF_6 was released without buoyancy from a tower at a height of 115 m and collection of tracer sampling units at the ground level positions at the maximum of three crosswind arcs. The sampling units were located at a distance of 2 to 6 km from the point of release. The site was mainly residential with a roughness length of 0.6 m.

3. Results and discussion

The meteorological parameters during the Copenhagen experiment are presented in Table 1. The values of ground-level centerline concentrations measured during Copenhagen experiment and the corresponding proposed values by the new model given by Eqn. (4) are presented in Table 2. Notice that the values of concentrations measured and predicted are normalized by the source strength (C/ Q).

A comparison between the normalized centerline ground-level concentrations observed and proposed by the new model as a function of downwind distance (x) for a orders of the derivative (α) are represented graphically as in Fig. 1. Fig. 1 shows the best value of the proposed model with observation concentration at $\alpha = 0.95$ than others values.

TABLE 2

Normalized observed (C_o / Q) and proposed (C_p / Q) centerline ground-level concentrations (10^{-7} sm^{-3}) for Copenhagen experiment considering different fractional parameters (α)

Exp.	Distance(m)	C_o/Q	C_p/Q		
			α = 0.9	α = 0.95	α = 1.0
1	1900	10.5	5.9	10.2	7.1
1	3700	2.14	2,56	2.18	2.03
2	2100	9.85	7.17	8.6	7.92
2	4200	2.83	2.45	2.58	2.16
3	1900	16.33	13.27	16.18	15.27
3	3700	7.95	6.43	7.05	6.44
3	5400	3.76	2.99	3.37	3.07
4	4000	15.71	13.78	15.66	14.06
5	2100	12.11	9.39	12.03	11.47
5	4200	7.24	6.39	7.85	8.45
5	6100	4.75	5.81	4.45	5.56
6	2000	7.44	6.48	7.14	6.36
6	4200	3.47	3.92	3.67	3.48
6	5900	1.74	1.61	1.78	1.71
7	2000	9.48	8.10	9.40	8.98
7	4100	2.62	2.00	2.24	2.20
7	5300	1.15	1.35	1.16	1.68
8	1900	9.76	8.79	9.72	9.32
8	3600	2.64	2.70	2.80	2.46
8	5300	0.98	0.86	0.92	0.91
9	2100	8.52	7.14	8.62	7.84
9	4200	2.66	2.27	2.71	2.59
9	6000	1.98	1.55	1.86	1.74

A scatter diagram of the normalized centerline ground-level observed concentrations against the corresponding values proposed by the new model for different orders of the derivative (α) is shown in Fig. 2. Also this figure shows that all the proposed model at α = 0.95 lies one to one line but the others values are located inside a factor of two.

3.1. *Statistical evaluation of the present model*

To evaluate the model accuracy, we used the following statistical idiocies that characterize the agreement between the predicted and observed concentrations. These measures are discussed by Hanna (1989) and defined as:

$$\text{Fraction Bias (FB)} = \frac{\left(\overline{C}_o - \overline{C}_p\right)}{\left[0.5\left(\overline{C}_o - \overline{C}_p\right)\right]}$$

Normalized Mean Square Error (NMSE)

$$= \frac{\overline{\left(C_p - C_o\right)^2}}{\left(\overline{C_p C_o}\right)}$$

Correlation Coefficient (COR)

$$= \frac{1}{N_m}\sum_{i=1}^{N_m}\left(C_{pi} - \overline{C_p}\right) \times \frac{\left(C_{oi} - \overline{C_o}\right)}{\left(\sigma_p \sigma_o\right)}$$

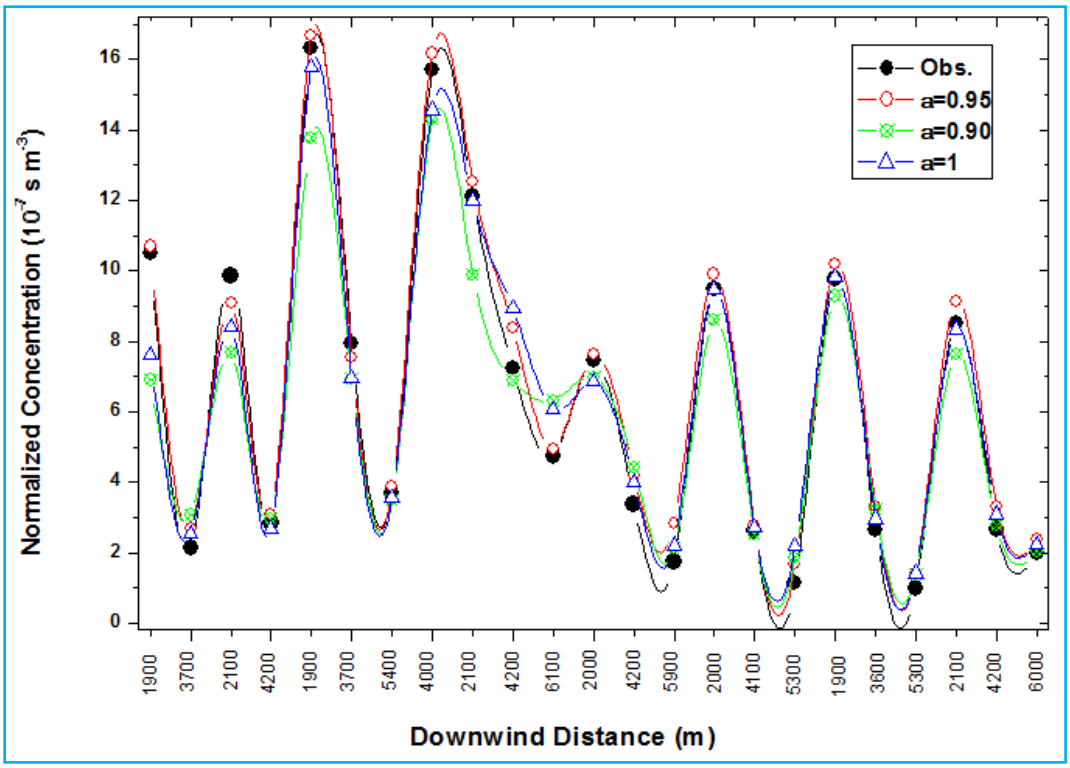

Fig. 1. Comparison between the normalized centerline ground-level concentrations observed and proposed as a function of downwind distance (x) for different orders of the derivative (α)

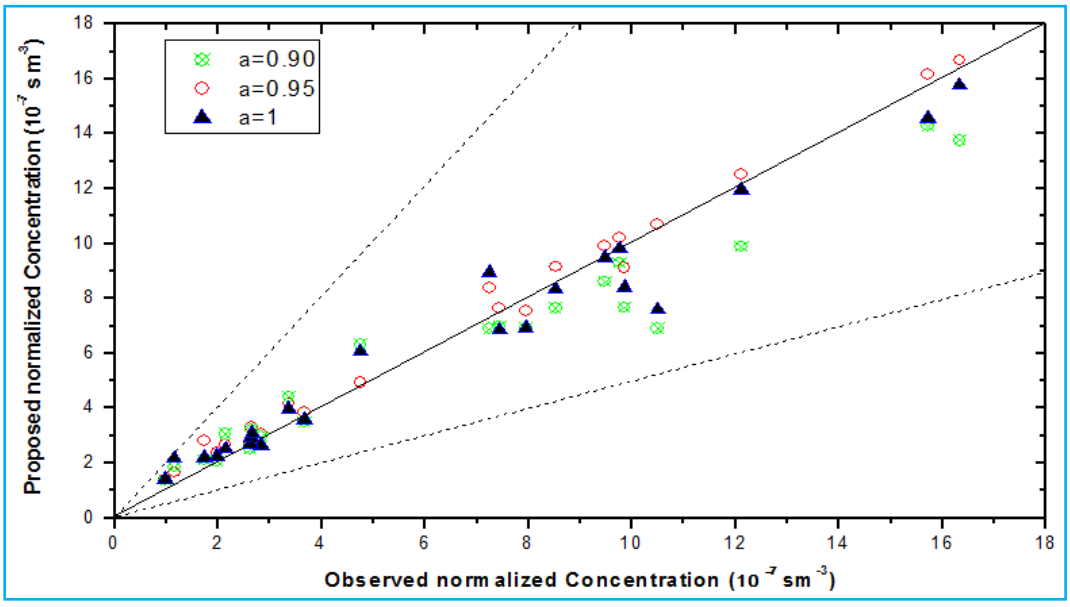

Fig. 2. A scatter diagram of the normalized concentrations observed and proposed by the present model for different orders of the derivative (α). Dotted lines indicate a factor of two; the solid line is the one-to-one line

$$\text{Factor of Two (FAC2)} = 0.5 \leq \frac{C_p}{C_o} \leq 2.0$$

where, σ_p and σ_o are the standard deviations of the proposed concentrations C_p and observed C_o respectively. The over-bar indicates the average value. The perfect

TABLE 3

Statistical evaluation of the proposed model for different values of α against Copenhagen experiment

α	NMSE	FB	COR	FAC2
0.90	0.08	0.17	0.97	0.89
0.95	0.00	0.02	1.0	0.97
3	0.03	0.09	0.98	0.94

model must have the following performances: NMSE = FB = 0 and COR = FAC2 = 1.0.

The calculated values of these statistical measures that evaluate the performance of the proposed model for different values of the order α of the derivative are shown in Table 3.

Table 2 and Fig. 1 show a very good agreement between the observed and proposed concentrations by the new model for the order of the derivative α = 0.95 and good agreement with α = 1.0 than α = 0.90.

Fig. 2 reveals that all the proposed concentrations by the new model with different orders (α) lie within a factor of two. Most of the proposed concentrations with α = 0.95 nearly locate on a perfect one to one line where the measured and the corresponding proposed concentrations are equal. This indicate that the best performance of the new model occurs when α = 0.95.

The values of the statistical measures in Table 3 show also a very good agreement between the observed and proposed concentrations by the derived model for α = 0.95 with NMSE = 0.0, FB = 0.02, COR = 1.0 and FAC2 = 0.97. The statistical values indicate that the best performance of the new model occurs when α = 0.95.

4. Conclusions

In this study we proposed an analytical treatment for the three-dimensional ADE in (*x*, *y* and *z* directions) of fractional order α (non-integer order) using the (GILTT) which is still good method and adopting Caputo' formula of a fractional derivative to simulate the dispersion of contaminants in the atmospheric boundary layer (ABL). Considering through this treatment that the wind speed *u* and the vertical eddy diffusivity K_z are functions of *z* only but the lateral eddy diffusivity K_y is function of (*x*, *z*). The accuracy of the proposed model for different values of α was evaluated against the Copenhagen experiment. The new model and the statistical analysis that performed between the proposed and experimental concentrations reveals that the best results were obtained with order α = 0.95 and good agreement with α = 1.0 than α = 0.90.

Disclaimer : The contents and views expressed in this study are the views of the authors and do not necessarily reflect the views of the organizations they belong to.

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