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On the numerical solution of diabatic quasi-geostrophic omega equation

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ABSTRACT. The quasi-geostrophic omega equation has been numerically solved to get the vertical velocity distribution in a typical westerly disturbance. The effects of sensible heat and latent heat of condensation are also included. Three dimensional relaxation was performed to get the numerical solution of the omega equation for a 4-layer model. The computations were performed on HITAC 5020.

The numerically obtained vertical velocity field is in good agreement with the observed weather pattern associated with the middle latitude large-scale disturbance, *i.e.*, ascending motion in front of the trough and downward motion in the rear of the trough.

1. Introduction

The problem of the vertical velocity computation has been of vital concern to meteorologists. Being small in magnitude but at the same time being important for weather and weather systems, especially in the study of ageostrophic divergent motion and energy transformation, many attempts have been made to find an accurate method of computing the vertical velocity. Therefore, in addition to kinematic and adiabatic methods, today we have sophisticated multilevel dynamical models to compute vertical velocity. Subsequent to a classical paper by Charney (1947), the quasi-geostrophic system of equations have been widely used. Although the system was evolved on the basis of its applicability to large scale systems of middle latitudes, and at present there is no rigorous justification for its applicability to the tropics, some recent studies (Hawkins 1967, Rao 1970, etc) indicate that the vertical motion obtained from the quasi-geostrophic omega equation is a fairly good approximation even in the tropics.

In the present study, an attempt has been made to solve the quasi-geostrophic omega equation numerically. The finite difference form and computational scheme etc have been elaborately described.

Diabatic forcing due to latent heat and sensible heat have been also introduced and the vertical velocity induced by these diabatic factors has been computed. In the present study, frictional effects have been neglected.

As a test experiment, the program for a numerical solution of the omega equation has been run on the data generated by another program for an analytical pattern, which simulates the features of a typical middle latitude westerly disturbance. The reason for taking a typical middle latitude westerly disturbance pattern is that, at present, we have sufficient observational information regarding the structure and associated vertical velocity distribution of such disturbances. The vertical velocities obtained are in good agreement (in magnitude and in spatial distribution) with the observed weather pattern associated with middle latitude large scale disturbances, *i.e.*, we find ascending motion in front of the trough and downward motion in the rear of the trough.

2. Equations governing the model

2.1. *Quasi-geostrophic system of equations*

Under the usual approximation based on the scale considerations by Charney (1947), the vorticity and thermodynamic energy equations may be written for frictionless and adiabatic motion as follows —

$$\frac{\partial \zeta}{\partial t} + V_g \cdot \nabla \eta = f_0 \frac{\partial \omega}{\partial p} \qquad (2.1)$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right) + V_g \cdot \nabla\left(-\frac{\partial \phi}{\partial p}\right) - S\omega = 0 \qquad (2.2)$$

where, $$S = -\frac{\partial \ln \theta}{\partial p} \qquad (2.3)$$

The omega equation is derived from these two equations by eliminating the time dependent term and by using the geostrophic relation. However, if the time dependent term is eliminated by use of a balance equation, the resulting diagnostic equation in ω will be a balanced baroclinic omega equation.

In accordance with the scale theory of Charney (1947) and energy considerations illustrated by Lorenz (1960), the advecting wind ($\mathbf{V}_g$) is taken to be the nondivergent geostrophic wind and f is taken as constant on the right hand side of (1). ζ is given by the expression —

$$\zeta = \mathbf{K} \cdot \nabla \times \mathbf{V}_g \tag{2.4a}$$

where $\mathbf{K}$ is the unit vector along the vertical axis. The constancy of f and g will be discussed later.

Operating equation (1) by $\partial/\partial p$ and equation (2) by $(1/f_0)\ \nabla^2$, we get after the substitution,

$$\nabla^2 \psi = \zeta \tag{2.4b}$$

$$\frac{\partial}{\partial p}\left\{\frac{\partial}{\partial t}\left(\nabla^2 \psi\right)\right\} + \frac{\partial}{\partial p}\left(V_g . \nabla \eta\right) = f_0 \frac{\partial^2 \omega}{\partial p^2} \tag{2.5}$$

$$\frac{1}{f_0} \nabla^2 \left\{\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right)\right\} + \frac{1}{f_0} \nabla^2 \left\{V_g . \nabla \left(-\frac{\partial \phi}{\partial p}\right)\right\}$$

$$-\frac{1}{f_0} \nabla^2 (S\,\omega) = 0 \tag{2.6}$$

where, ψ the geostrophic stream function is defined by —

$$f_0 \nabla^2 \psi = \nabla^2 \phi \tag{2.7}$$

When the time dependent term is eliminated, we get the following diagnostic equation in ω —

$$\nabla^2 \omega + \frac{f_0^2}{S} \frac{\partial^2 \omega}{\partial p^2} = \frac{1}{S}\left[f_0 \frac{\partial}{\partial p}\left(V_g . \nabla \eta\right) + \nabla^2 \left\{V_g . \nabla \left(-\frac{\partial \phi}{\partial p}\right)\right\}\right]$$

$$= \frac{1}{S}\left[\frac{\partial}{\partial p} J(\phi, \eta) + \frac{1}{f_0} \times \nabla^2 \left\{J\left(\phi, -\frac{\partial \phi}{\partial p}\right)\right\}\right] \tag{2.8}$$

The above equation was solved numerically by three dimensional relaxation. The details of the procedure will be discussed in a later section.

2.2. Variation of static stability parameter

Eq. (2·8) has been derived under the assumption that S is a function of p only and does not vary with x and y. In order to maintain energetic consistency, it is necessary that S should be function of p only (Sumner 1950, Wiin-Nielsen 1959, Lorenz 1960 and Saito 1960). This assumption makes it convenient to solve Eq. (2·8).

2.3. Variation of Coriolis parameter

It may be recalled that in (2.1), f is taken as f_0. This condition is imposed to satisfy the constraints of energy invariance and conservation of vorticity. Some additional terms should appear on L.H.S. to satisfy the constraints imposed by the invariance of kinetic energy and vorticity, if f is taken as a variable (Appendix I). However, it may be noted that f is taken as a variable for the purpose of computations of absolute vorticity. As we are not concerned in the prognostic aspects of the quasi-geostrophic model in this study, the problem of a variable f does not pose any serious problem here.

3. Quasi-geostrophic system of equations including diabatic forcing terms

Taking into consideration the nonadiabatic heating effects, the thermodynamic energy equation (2.2) may be written as —

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right) + V . \nabla\left(-\frac{\partial \phi}{\partial p}\right) - S\,\omega = \frac{R}{C_p . p} \frac{dQ}{dt} \tag{3.1}$$

where, dQ/dt is the nonadiabatic rate of heating per unit time and unit mass. Following the procedure given in Section 2, a new diagnostic equation may be derived given as —

$$\nabla^2 \omega + \frac{f_0^2}{S} \frac{\partial^2 \omega}{\partial p^2} = \frac{1}{S}\left[\frac{\partial}{\partial p} J(\phi, \eta) + \frac{1}{f_0} \times \nabla^2 \left\{J\left(\phi, -\frac{\partial \phi}{\partial p}\right)\right\} - \frac{R}{C_p . p} \nabla^2 \frac{dQ}{dt}\right] \tag{3.2}$$

Hereafter, we drop the subscript g for geostrophic motion.

As (3·2) is a linear equation in ω, it can be resolved into components —

$$\nabla^2 \omega_0 + \frac{f_0^2}{S} \frac{\partial^2 \omega_0}{\partial p^2} = \frac{1}{S}\left[\frac{\partial}{\partial p} J(\phi, \eta) + \frac{1}{f_0} \times \nabla^2 \left\{J\left(\phi, -\frac{\partial \phi}{\partial p}\right)\right\}\right] \tag{3.3}$$

$$\nabla^2 \omega^* + \frac{f_0^2}{S} \frac{\partial^2 \omega^*}{\partial p^2} = -\frac{1}{S} \frac{R}{C_p . p} . \nabla^2 \left(\frac{dQ}{dt}\right) \quad (3\cdot4)$$

where, $\omega = \omega_0 + \omega^*$ (3.5)

Because of the linearity of (3.2), solutions obtained by solving (3.3) and (3.4) separately may be added to get the final solution under appropriate boundary conditions.

dQ/dt, is the rate of non-adiabatic heating per unit time and unit mass, may be produced by latent heat of condensation, sensible heat transfer, radiation and friction. In the present study only the first two factors have been considered. Consequently, if dQ_L/dt and dQ_S/dt are the rate of heating per unit time and unit mass by latent heat of condensation and sensible heat supply respectively we have,

$$\frac{dQ}{dt} = \frac{dQ_L}{dt} + \frac{dQ_S}{dt} \quad (3.6)$$

Therefore, (3·4) being linear may be further split up into the following equations—

$$\nabla^2 \omega_1 + \frac{f_0^2}{S} \frac{\partial^2 \omega_1}{\partial p^2} = -\frac{1}{S} \frac{R}{C_p . p} \nabla^2 \left(\frac{dQ_L}{dt}\right) \quad (3.7)$$

$$\nabla^2 \omega_2 + \frac{f_0^2}{S} \frac{\partial^2 \omega_2}{\partial p^2} = -\frac{1}{S} \frac{R}{C_p . p} \nabla^2 \left(\frac{dQ_S}{dt}\right) \quad (3.8)$$

where, ω_1 and ω_2 are the vertical velocities due to the latent heat of condensation and sensible heat supply respectively. Consequently,

$$\omega^* = \omega_1 + \omega_2 \quad (3.9a)$$

$$\text{and} \quad \omega = \omega_0 + \omega_1 + \omega_2 \quad (3.9b)$$

Three dimensional relaxation will be performed separately to solve equations (3·3), (3·7) and (3·8) and the final ω will be given by the algebraic sum of ω_0, ω_1 and ω_2 obtained from equations (3·3), (3·7) and (3·8) respectively.

3.1. *Inclusion of the latent heat of condensation*

One of the obstacles in solving (3·7) is the parameterization of dQ_L/dt in terms of known or observable meteorological quantities. Here, we follow the method adopted by the electronic computation centre of Japan Met. Agency (1963).

For a saturated atmosphere,

$$\frac{dQ_L}{dt} = -L\frac{dq^*}{dt} \quad (3.1.1)$$

where, L is the latent heat of condensation (assumed to be constant) and q^* is the saturated specific humidity. Following Gambo (1963), it may further be written as,

$$\frac{dQ_L}{dt} = -L\frac{dq^*}{dt} = -\omega L F^* \quad (3.1.2)$$

where, F^* is the condensation rate and is function of p and T only. The exact mathematical expression for F^* is given as,

$$F^*(p, T) = \frac{1}{1 + \frac{L}{C_p}\left(\frac{\partial q^*}{\partial T}\right)_p} \left[\left(\frac{\partial q^*}{\partial p}\right)_T + \frac{XT}{p}\left(\frac{\partial q^*}{\partial T}\right)_p\right] \quad (3.1.3)$$

$$(X = R/C_p \approx 0.28)$$

In the above expression, subscript p or T denote the differentiation at p=const. respectively. The derivation of the expression for F^* is given in Appendix II.

However it is seen, that in (3.1.2), ω which is yet to be computed appears explicitly. Therefore ω_0 which is calculated by solving (3·3) is taken as the first approximation of ω. As a matter of fact, if ω_2 is also known independently, $(\omega_0 + \omega_2)$ may also be taken as a first approximation for ω. Since the formulation for dQ_S/dt is such that ω_2 can be found independently by solving (3·8), $(\omega_0 + \omega_2)$ has been taken as the first approximation of ω.

Let, $\omega' = \omega_0 + \omega_2$ (3·1·4)

Therefore using (3·1·2) and (3·1·4), (3·7) may be written as—

$$\nabla^2 \omega_1 + \frac{f_0^2}{S} \frac{\partial^2 \omega_1}{\partial p^2} = -\frac{1}{S} \frac{R}{C_p . p} \nabla^2 (-\omega' L F^*) \quad (3.1.5)$$

Let, $\dfrac{LF^*R}{C_p . p} = S^*$

It is interesting to note that unit of S^* ($m^2\,sec^{-2}\,mb^{-2}$) is same as that of S. Usually, (3·1·5) is written as,

$$\nabla^2 \omega_1 + \frac{f_0^2}{S} \frac{\partial^2 \omega_1}{\partial p^2} = \nabla^2 \left(\frac{S^*}{S}\omega'\right) \quad (3.1.6)$$

Saito (1960) has evaluated the approximate values of S^* at different isobaric surfaces for different ranges of temperatures. The mathematical expressions are shown in Appendix II.

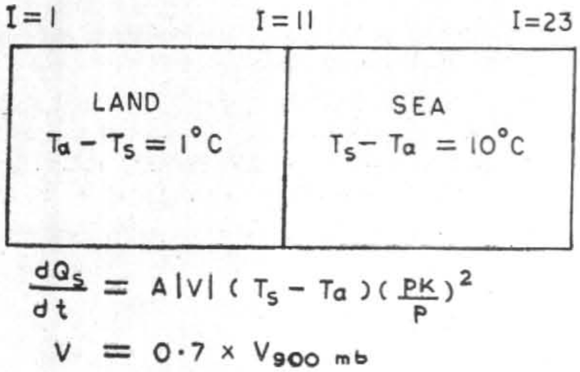


Fig. 1. Artificial land and sea distribution for calculating the sensible heat effect

In order that the solution of $(3\cdot 1\cdot 6)$ may converge, it is always necessary to satisfy the criteria,

$$S^* - S < 0 \qquad (3.1.7)$$

Wherever this criterion is not satisfied, S^* is forced to be equal to $0.8\,S$ or below. The details of this criterion for convergence are shown in Appendix III.

Normally we do not expect much precipitation to take place above 400 mb; consequently ω' is put equal to zero at or above 400 mb. For the levels below 400 mb, it is assumed that,

$$\frac{dQ_L}{dt} = 0 \qquad \text{if } \omega' > 0$$

$$\text{and} \quad \frac{dQ_L}{dt} = -\omega' LF^* \qquad \omega' < 0$$

3.2. *Inclusion of sensible heat*

The role of sensible heat supply has also been parameterized in the way similar to the one adopted in the operational 4-level quasi-geostrophic baroclinic model in Japan (1963).

Following Jacobs (1951), Martin (1962) and Spart (1962), the eddy flux of heat H per unit time into an air column of unit cross-section is computed by the relation,

$$H = AV(T_s - T_a) \qquad (3.2.1)$$

where, A is a constant, V is the surface wind speed, T_s is the surface water temperature and T_a is the surface air temperature. It is further assumed that the heating decreases with the the decreasing pressure according to a power law. Therefore, the rate of heating per unit time and unit mass, at the pressure level p is given by,

$$\frac{dQ_S}{dt} = A\,|V|\,(T_s - T_a)\left(\frac{p}{p^*}\right)^{\gamma}$$

p^* is surface pressure and A, γ are constants. The constant A has been given the numerical value,

$$A = 1.0 \times 10^{-3}\,\text{m sec}^{-1}\,\text{deg}^{-1}$$

Following the suggestion by Manabe (1958) that rate of heating due to the sensible heat supply decreases rapidly with height, γ has been taken as 2. It may be easily seen that the order of of magnitude of dQ_S/dt is comparable to the order of magnitude of dQ_L/dt for the normally observed surface winds, provided the temperature difference between surface and the overlying air is of the order of $1 \sim 10°C$.

On physical grounds, upward motion may be expected if there is sensible heat supply from an ocean surface to the overlying air, but a downward current may not be obviously expected if sea surface is colder than the overlying air. Therefore, assuming downward flux of sensible heat to be small, tentatively, if $T_a > T_s$

$$A = 1.0 \times 10^{-4}\,\text{m sec}^{-1}\,\text{deg}^{-1}$$

is taken in this study. This reduction in the value of A will not allow considerable descending motion even if sea surface is colder than the overlying air.

Sensible heat exchange between the atmosphere and continent has not been considered. For the purpose of the present study, an artificial land sea distribution was made (as shown in Fig. 1) and a pre-determined temperature contrast was imposed between the land-sea interface and atmosphere-ocean interface.

4. Generation of the input data

As mentioned earlier, a typical westerly disturbance superimposed over the basic zonal westerly current of pre-specified wind shear has been taken to obtain the associated vertical velocity distribution by numerical integration of the omega equation.

If $\bar{\phi}$ is the geopotential field corresponding to the basic zonal current and ϕ' is the geopotential field for the perturbation, the field ϕ, which has been used for our computation, is given by—

$$\phi = \bar{\phi}\,(y, p) + \phi' \qquad (4.1)$$

The values of zonal westerly wind speeds which have been taken for the present computations are shown in Table 1. Temperatures for a standard atmosphere have been also given for each pressure level.

TABLE 1

Vertical structure of the model atmosphere taken for the computations

k	Pressure (mb)	Temperature for standard atmosphere (°A)	Zonal wind speed (km/hr)
0	200	217·0	130
1	300	228·0	115
2	400	241·0	100
3	500	252·0	85
4	600	261·2	70
5	700	268·5	55
6	800	275·5	40
7	900	281·5	25
8	1000	287·0	10

Using the geostrophic relation and the equation of state, we have,

$$\frac{\partial \bar{u}}{\partial p} = \frac{R}{fp} \frac{\partial T}{\partial y} \tag{4.2}$$

Knowing $\partial \bar{u}/\partial p$ and prescribing the value of T along the middle of the domain from the above table, $T(y, p)$ can be found for all the levels. From $T(y,p)$ the hydrostatic relation,

$$\frac{\partial \bar{\phi}}{\partial p} = -\frac{R\bar{T}}{p} \tag{4.3}$$

may be integrated with the boundary condition $\bar{\phi} = 0$ for $k = 8$. This choice of the boundary condition is arbitrary. However, any constant value of $\bar{\phi}$ will also give the same ω field.

$$\text{Therefore,} \quad \bar{\phi} = -\int \frac{R\bar{T}}{p}\, dp \tag{4.4}$$

Since $T(y, p)$ is known by (4·2), $\bar{\phi}(y,p)$ may be calculated from (4·4).

In order to compute ϕ', a sinusoidal disturbance of wavelength 6500 km was considered. In order that there may not be any inflow or outflow from the northern and southern boundary, the disturbance is made to vanish along the northern and southern boundaries. Therefore, if D is the width of the computation domain, the analytical form of a sinusoidal disturbance having a tilt μ along the vertical is,

$$\phi' = A_m \sin \frac{2\pi}{\bar{L}} (x - \mu) \sin \frac{\pi}{D} y \tag{4.5}$$

where $\bar{L}$ is the wavelength of the disturbance and A_m is the amplitude.

It may be noticed that μ is a function of p only. μ has been taken to be half the unit grid interval d ($d = 250$ km) for 100 mb. Since the domain considered for the computations is (23×17), *i. e.*, 23 and 17 grid points along x and y respectively the value of D is,

$$D = 16 \times d = 4000 \text{ km}$$

Differentiating (4·5) w. r. to x we have,

$$\frac{\partial \phi'}{\partial x} = A_m \frac{2\pi}{\bar{L}} \cos \frac{2\pi}{\bar{L}} (x - \mu) \sin \frac{\pi}{D} y$$

By the geostrophic relation, therefore,

$$f.v' = A_m \frac{2\pi}{\bar{L}} \left\{ \cos \frac{2\pi}{\bar{L}} (x - \mu) \sin \frac{\pi}{D} y \right\}$$

Therefore, $f(v'_{max}) = A_m (2\pi/\bar{L})$, because the maximum possible value of the expression under curly brackets is one. Specifying, therefore, the value of v'_{max}, A_m can be evaluated. In the present computation, v'_{max} has been taken as 5 m sec^{-1}.

Since,

$$A_m = \frac{\bar{L}.f.v'm}{2\pi} \tag{4.6}$$

Eq. (4·5) may be written as,

$$\phi' = \frac{\bar{L}.f.v'm}{2\pi} \sin \frac{2\pi}{\bar{L}} (x - \mu) \sin \frac{\pi}{D} y \tag{4.7}$$

Now, since all the variables on R. H. S. have been specified ϕ' may be computed for different values of x and y. The amplitude and wavelengths of ϕ' has been taken as constant along the vertical, therefore, (4·7) holds for all the levels.

Thus, adding (4·4) and (4·7), ϕ can be computed. Since the finite difference forms of these equations are straightforward, they will not be discussed separately.

The contour pattern of geopotential for the 300 mb surface has been shown in Fig. 2. The vorticity distribution for the corresponding level is shown in Fig. 3. For economy in space, contours and vorticity patterns for other levels have not been presented.

5. Finite difference forms of the differential equations and numerical computation

The finite difference forms of the differential equations which appear in Sections 2 and 3,

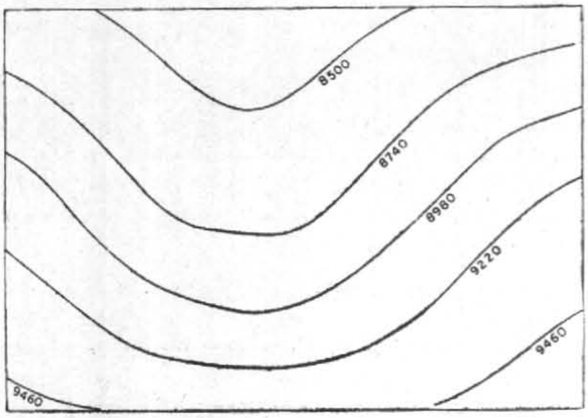


Fig. 2. Initial height field for level $k=1$ (300 mb)

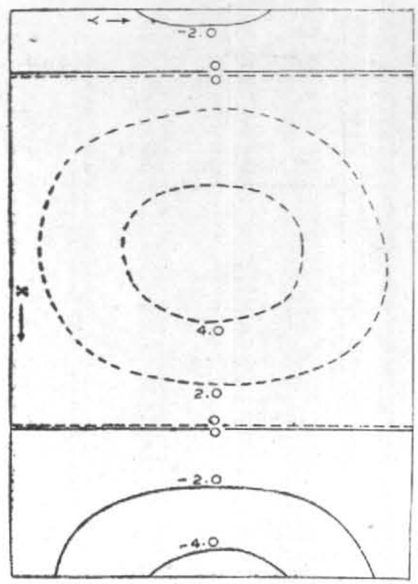


Fig. 3. Vorticity field for level $k=1$ (300 mb)

Unit : 10^{-5} sec^{-1}

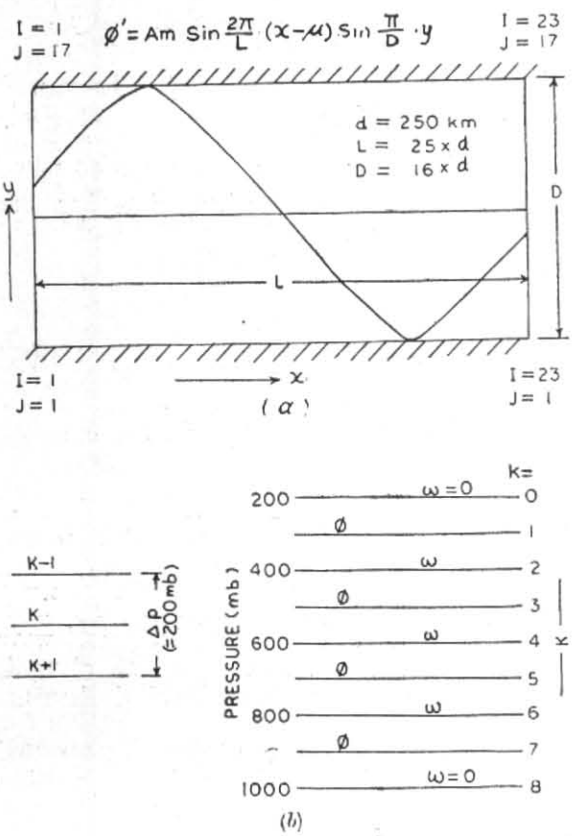


Fig. 4

suitable for numerical coumputation, will be presented in this section. The vertically staggared grid, which has been used for the computation of vertical derivative has been shown in Fig. 4(b). The number of grid points in the x and y directions, and the corresponding grid interval has also been shown in Fig. 4(a) . It may be seen from Fig. 4(b) that, we need to know the information regarding height at the levels $k=1$, 3, 5 and 7 and knowing ω at $k=0$ and 8 (as boundary condition), ω at levels $k=2$, 4 and 6 may be computed by the three dimensional relaxation of the relevant equation. Finite difference forms suitable for numerical computation of static stability parameter S, will be discussed separately.

For a square grid system as shown in Fig. 4(c), of grid interval d the expression for Laplacian of the variable ϕ is,

$$\nabla^2\phi = \frac{m^2}{d^2}\left(\phi_1+\phi_2+\phi_3+\phi_4-4\phi_0\right) \qquad (5.1)$$

For the sake of convenience, hereafter, we shall use the notation $\underset{\sim}{\nabla}^2\phi$ for the expression under parentheses. Considering the map factor to be unity (*i.e.*, $m=1$),

$$\left.\begin{aligned} \nabla^2\phi &= \frac{1}{d^2}\underset{\sim}{\nabla}^2\phi \\ \text{Similarly,}\quad J(a,b) &= \frac{1}{d^2}\underset{\sim}{J}(a,b) \end{aligned}\right\} \qquad (5.2)$$

Following these notations the finite difference form of (2·8) which was used for computation is,

$$\begin{aligned} \underset{\sim}{\nabla}^2\,\omega_k + \frac{d^2f^2}{S_k(\triangle p)^2}\left(\omega_{k-2}+\omega_{k+2}-2\omega_k\right) \\ = \frac{g}{S_k\,\triangle p}\left\{\underset{\sim}{J}(\eta,Z)_{k-1} - \underset{\sim}{J}(\eta,Z)_{k+1}\right\} \\ - \frac{g^2}{S_k f d^2\,\triangle p}\,\underset{\sim}{\nabla}^2\left[\underset{\sim}{J}\left\{Z_{k-1},\ Z_{k+1}\right\}\right] \end{aligned} \qquad (5.3)$$

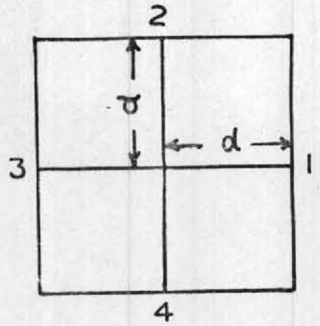


Fig. 4(c)

Because,

$$J\left\{\left(Z_{k-1}-Z_{k+1}\right),\left(\frac{Z_{k+1}+Z_{k-1}}{2}\right)\right\}=\underset{\sim}{J}(Z_{k-1}, Z_{k+1})$$

As the vorticity is calculated only for the points internal to the outer boundary, the value of vorticity is needed for the points on the outer boundary to calculate the Jacobian $[J(\eta, Z)]$. The absolute vorticity for any point on the boundary was made equal to the Coriolis parameter at that point if there is inflow in the domain across that point and was set equal to the absolute vorticity at the adjacent grid point if there was outflow . This criterion was adopted to have partial control over undesirable inflow and outflow of vorticity from the computation domain.

Similarly, we also need the value of Jacobian $[J(\phi_1, \phi_2)]$ at the outer boundary in order to compute its Laplacian at the points internal to the outer boundary. The values of the Jacobian at the adjacent points has been taken, in the present study, as the value of Jacobian at the outer boundary points.

5.2. *Computation of static stability parameter*

The static stability parameter S, is given by the expression,

$$S=-\frac{\alpha}{\theta}\frac{\partial \ln \theta}{\partial p} \tag{5.2.1}$$

Taking the logarithmic differentiation of Poisson's equation and using the equation of state and hydrostatic relation, the above equation may be written as,

$$S=\frac{\partial^2\phi}{\partial p^2}+\frac{C_v}{C_p}\frac{1}{p}\frac{\partial\phi}{\partial p} \tag{5.2.2}$$

Referring to the vertical grid in Fig. 4(b), the finite difference form of (5·2·2) may be written as,

$$S_k=g\left[\left\{\left(Z_{k+1}+Z_{k-1}-2Z_k\right)\Big/(\triangle p)^2\right\}-\right.$$
$$\left.-\frac{C_v}{C_p}\frac{1}{p_k}\left\{\left(Z_{k-1}-Z_{k+1}\right)\Big/(2\triangle p)\right\}\right] \tag{5.2.3}$$

where, $\triangle p = p_k - p_{k-1} = p_{k+1} - p_k$

In the actual computations, $\triangle p$, was taken as 200 mb. As information on Z is available for levels $k = 1, 3, 5$ and 7 the above formulation will enable us to get S at $k=3$ and 5. From these two values, S is obtained for the levels $k=2$, 4 and 6 with suitable weighting functions.

The following interpolation was used—

$$\left.\begin{aligned} S_2 &= S_3+(S_3-S_5)/2\\ S_4 &= (S_3+S_5)/2\\ S_6 &= S_4\,(600/800)^2 \end{aligned}\right\} \tag{5.2.4}$$

As discussed in Section 2·2, S has been taken to be a function of pressure only. For any level, the value of S is taken to be constant and the appropriate value for that level is,

$$S_k=\frac{1}{(M\times N)}\left|\sum_{j=1}^{N}\sum_{i=1}^{M}S_{ij}\right|_k \tag{5.2.5}$$

where, M and N are respectively the number of grid points along the x and y directions.

5.3. *Finite difference form of the omega equation involving diabatic forcing*

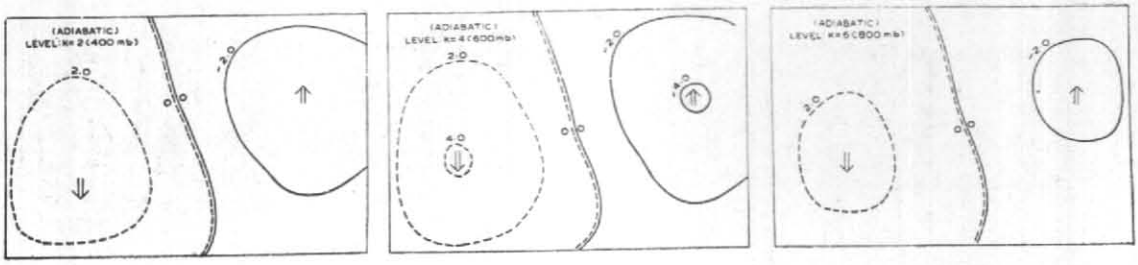
Following the procedure adopted in Sec. 5·1, the finite difference form for the equations (3·8) and (3·1·6) may be written as follows—

$$\underset{\sim}{\nabla}^2(\omega_2)_k+\frac{f^2d^2}{S_k(\triangle p)^2}\left\{(\omega_2)_{k-2}+(\omega_2)_{k+2}-2(\omega_2)_k\right\}$$
$$=-\frac{R}{C_p\cdot p}\frac{1}{S_k}\underset{\sim}{\nabla}^2\left(\frac{d_{\theta s}}{dt}\right)_k \tag{5.3.1}$$

$$\underset{\sim}{\nabla}^2(\omega_1)_k+\frac{f^2d^2}{S_k(\triangle p)^2}\left\{(\omega_1)_{k-2}+(\omega_1)_{k+2}-2(\omega_1)_k\right\}$$
$$=\underset{\sim}{\nabla}^2\left(\frac{S^*}{S}\,\omega'\right) \tag{5.3.2}$$

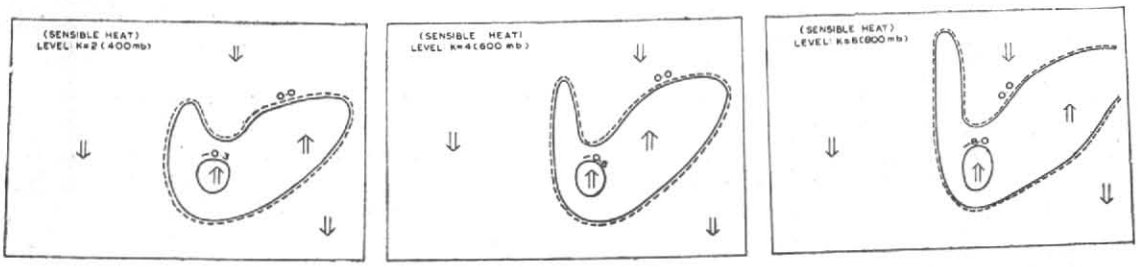
It may be recalled that—

$$S^*=\frac{LR}{C_p\cdot p}F^*$$



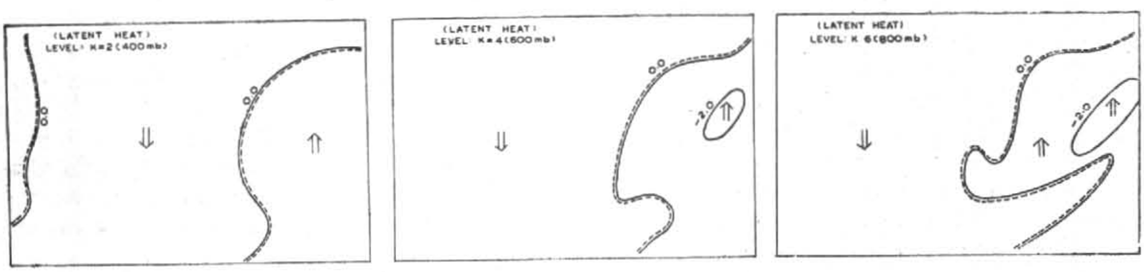
(a) (b) (c)

Fig. 5. Vertical velocity field (adiabatic) at level (a) $k=2$(400 mb) (b) $k=4$ (600 mb) (c) $k=6$ (800 mb)
Unit : mb/hr



a) (b) (c)

Fig. 6. Vertical velocity field (sensible heat) at level (a) $k=2$ (400 mb) (b) $k=4$ (600 mb) (c) $k=6$(800 mb)
Unit: mb/hr



(a) (b) (c)

Fig. 7. Vertical velocity field (latent heat) at level (a) $k=2$ (400 mb) (b) $k=4$ (600 mb) (c) $k=6$ (800 mb)
Unit : mb/hr

Following Saito (1960) the expressions given below were used for the computation of S^* for $k=2$, 4 and 6.

$$\left.\begin{aligned}(S^*)_2 &= \{25.0 + (T_2 - 249.0) \times 2.0\ \} \times 10^{-3}\\ (S^*)_4 &= \{\ 8.0 + (T_4 - 249.0) \times 0.81\} \times 10^{-3}\\ (S^*)_6 &= \{\ 3.0 + (T_6 - 249.0) \times 0.46\} \times 10^{-3}\end{aligned}\right\} \qquad (5.3.3)$$

The forcing due to latent heat of condensation was made to vanish at $k=2$, because condensation at 200 mb is negligible.

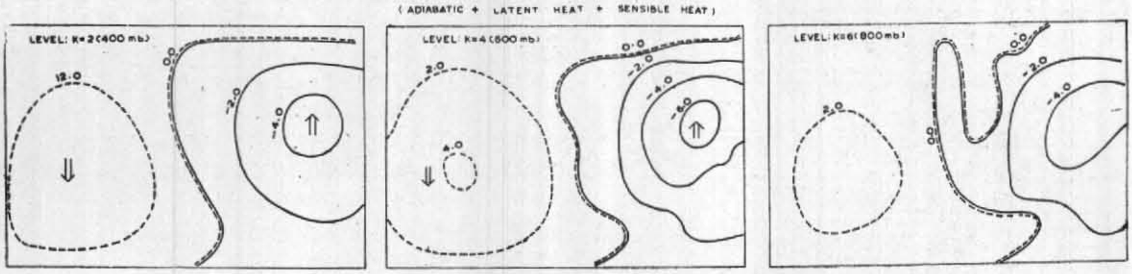
The approximation relations given as (5·3·3) hold for the temperature range of 249° C to 286°C. From the temperature profile S^* was calculated at every grid point.

5.4. *Three dimensional relaxation*

We used the accelerated Liebman relaxation technique, with an over-relaxation coefficient of 0·3. The tolerance for the residual of vertical velocity was taken as 0·00001 mb/sec.

6. Results

Figs. 5(a), 5(b) and 5(c) give the vertical velocity distribution due to differential vorticity advection, and the Laplacian of thermal advection. A detailed study of these vertical velocity fields reveal a westward displacement with height in the centres of ascending (descending) motion ahead (rear) of the trough. This is because of the fact that the westward tilt was prescribed while specifying the vertical structure of the disturbance.



(a) (b) (c)

Fig. 8. Vertical velocity field (adiabatic+latent hea'+sensible heat) at level (a) k=2 (400 mb) (b) k=4 (600 mb) and (c) k=6 (800 mb)

Unit : mb/hr

This is also commonly noticed on synoptic charts. However, the displacement in the present case was only by one grid length (250 km) at 200 mb. This is so because the initial disturbance also had the same tilt.

Figs. 6(a), 6(b), 6(c) and 7(a), 7(b), 7(c) provide the vertical velocity distribution caused by sensible heat and latent heat of condensation. It may be seen that ascending motion due to sensible heat is found only in those regions where sea surface is warmer than the overlying air. This factor may be of importance only in those cases where sea surface is sufficiently warmer than the overlying air. Such a situation may arise in the case of polar out-break when very cold air spreads over the warm oceanic waters. The role of latent heat of condensation, if introduced in the way being done as in the present study, is to enhance the magnitude of the ascending motion in those regions where ascending motion already exist due to vorticity advection and the Laplacian of thermal advection.

Figs. 8(a), 8(b) and 8(c) give the vertical velocity distribution due to the combined forcings on the right hand side of equation (3·2). It may be recalled that the vertical velocities given in Fig 8 are the algebraic sum of the vertical velocities in Figs. 5, 6 and 7.

It is seen that the magnitudes of the vertical velocities and also the spatial distribution is in agreement with the observed weather distribution associated with westerly disturbances of middle latitude. Ascending motion in front of the trough and descending motion in the rear is readily inferred from the given figures.

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APPENDIX I

Variability of f

Recalling (2·1), we have,

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla \eta = f \frac{\partial \omega}{\partial p} \tag{A.I.1}$$

If we integrate for the whole domain, we have,

$$\iiint \frac{\partial \zeta}{\partial t} + \iiint V \cdot \nabla \eta = \iiint f \frac{\partial \omega}{\partial p}$$

In order that vorticity and kinetic energy are conserved, f should be constant. However, if f is taken as variable, some additional terms should be included in order to satisfy the conservation criterion, and the equation should be taken in the following form—

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla \eta + V_\psi \cdot \nabla f = f \frac{\partial \omega}{\partial p} \tag{A.I.2}$$

where, $V = V_\psi + V_\chi$

APPENDIX II

Calculation of condensation rate, F^*

Recalling (3·1·2), we have,

$$\frac{dq^*}{dt} = \omega F^* \tag{A.II.1}$$

$\frac{dq^*}{dt}$ may be written as,

$$\frac{dq^*}{dt} = \frac{\partial q^*}{\partial t} + V \cdot \nabla q^* + \omega \cdot \frac{\partial q^*}{\partial p} \tag{A.II.2}$$

At a constant pressure surface, q^* is function of temperature only.

Hence, $$\partial q^* / \partial t = (\partial q^* / \partial T)(\partial T / \partial t) \tag{A.II.3}$$

We also know that—

$$\ln \theta e = \ln \theta + \frac{Lc}{C_p . T_c} q^*$$

where, subscript c refers to condensation level. In moist adiabatic ascent, the equivalent potential temperature is conserved; whence,

$$\frac{d}{dt} (\ln \theta e) = 0$$

The above expression may be expanded (dropping the subscript c) to,

$$\frac{\partial}{\partial t} (\ln \theta) + V \cdot \nabla (\ln \theta) + \omega \frac{\partial}{\partial p} \ln \theta = - \frac{L}{C_p . T} \frac{dq^*}{dt} \tag{A.II.4}$$

On an isobaric surface,

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) \ln \theta = \left(\frac{\partial}{\partial t} + V \cdot \nabla\right) \ln T \qquad \text{(A.II.5)}$$

Therefore, (A. II. 4) may be written as,

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) T + T\omega \frac{\partial \ln \theta}{\partial p} = -\frac{L}{C_p}\frac{dq^*}{dt} \qquad \text{(A.II.6)}$$

Making the substitution (A. II. 3), (A. II. 6) may be written as,

$$\frac{\partial T}{\partial t} + V \cdot \nabla T + \left(\omega \frac{\partial}{\partial p} (\ln \theta)\right) T + \frac{L}{C_p}\left\{\frac{\partial q^*}{\partial T}\left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T\right) + \omega \frac{\partial q^*}{\partial p}\right\} = 0$$

Therefore, arranging the terms we have,

$$\frac{\partial T}{\partial t} = -\mathbf{V} \cdot \nabla T - \omega(a)\,(b)^{-1} \qquad \text{(A.II.7)}$$

where, $(a) = \left(\frac{T}{\theta}\frac{\partial \theta}{\partial p} + \frac{L}{C_p}\frac{\partial q^*}{\partial p}\right)$ and $(b) = \left(1 + \frac{L}{C_p}\frac{\partial q^*}{\partial T}\right)$

Making the substitutions, (A. II. 3) and (A.II.7) in (A.II.2) we have,

$$\frac{\partial q^*}{\partial t} = \frac{\partial q^*}{\partial T}\left\{-V \cdot \nabla T + V \cdot \nabla T - \omega(a)\,(b)^{-1}\right\} + \omega \frac{\partial q^*}{\partial p}$$

$$\frac{dq^*}{\partial t} = \omega \left\{\frac{q^*}{\partial p} - \frac{\partial q^*}{\partial T}(a)\,(b)^{-1}\right\} = \frac{\omega}{(b)}\left|\frac{\partial q^*}{\partial p} - \frac{\partial q^*}{\partial T}\frac{T}{\theta}\frac{\partial \theta}{\partial p}\right| \qquad \text{(A.II.8)}$$

In the above expression, $\partial q^*/\partial p$ denotes the variation of q^* as the particle goes across the constant pressure surface. But, as we know that in this process, the particle experiences not only the variation of pressure but also the temperaure, therefore, we may write,

$$\frac{\partial q^*}{\partial p} = \left(\frac{\partial q^*}{\partial p}\right)_T + \left(\frac{\partial q^*}{\partial T}\right)_p \frac{\partial T}{\partial p} \qquad \text{(A.II.9)}$$

where, subscript p or T refers to the differentiation for $p =$ constant and $T =$ constant.

By differentiating (logarithmically), the equation

$$\theta = T\left(\frac{p_0}{p}\right)^{R/C_p}$$

we have,

$$\frac{T}{\theta}\frac{\partial \theta}{\partial p} = \frac{\partial T}{\partial p} - \frac{R}{C_p}\frac{T}{p} \qquad \text{(A.II.10)}$$

Making the substitutions (A.II.9) and (A.II.10) in (A.II.8) we have,

$$\frac{dq^*}{dt} = \frac{\omega}{(b)}\left[\left(\frac{\partial q^*}{\partial p}\right)_T \left(\frac{\partial q^*}{\partial T}\right)_p \frac{\partial T}{\partial p} - \left(\frac{\partial q^*}{\partial T}\right)_p \left(\frac{T}{\theta}\frac{\partial \theta}{\partial p}\right)\right]$$

$$= \frac{\omega}{(b)}\left[\left(\frac{\partial q^*}{\partial p}\right)_T + \frac{R}{C_p}\frac{T}{p}\left(\frac{\partial q^*}{\partial T}\right)_p\right]$$

Recalling (A.II.1), we have,

$$F^* = \left\{1 + \frac{L}{C_p}\left(\frac{\partial q^*}{\partial T}\right)^{-1}\right\}\left[\left(\frac{\partial q^*}{\partial p}\right)_T + \frac{R}{C_p}\frac{T}{p}\left(\frac{\partial q^*}{\partial T}\right)_p\right]$$

Saito (1960) has calculated the approximate value of the expression $(LRF^*/C_p.p)$ for different ranges of temperature at different isobaric levels. The approximate expressions are given below —

$$\left.\begin{aligned}\frac{LRF^*}{C_p.p} &= \{\,17.0 + (T_4 - 244.0) \times 0.18\,\} \times 10^{-3} \\ &= \{\,5.0 + (T_6 - 244.0) \times 0.6\,\} \times 10^{-3} \\ &= \{\,2.5 + (T_8 - 244.0) \times 0.3\,\} \times 10^{-3}\end{aligned}\right\}\ \text{for } 244 < T(k) < 249$$

$$\left.\begin{aligned}\frac{LRF^*}{C_p.p} &= \{\,25.0 + (T_4 - 249.0) \times 2.0\,\} \times 10^{-3} \\ &= \{\,8.0 + (T_6 - 249.0) \times 0.81\,\} \times 10^{-3} \\ &= \{\,3.0 + (T_8 - 249.0) \times 0.46\,\} \times 10^{-3}\end{aligned}\right\}\ \text{for } 249 < T(k) < 286$$

$$\left.\begin{aligned}\frac{LRF^*}{C_p.p} &= \{\,99.0 + (T_4 - 286.0) \times 1.23\,\} \times 10^{-3} \\ &= \{\,58.0 + (T_6 - 286.0) \times 0.70\,\} \times 10^{-3} \\ &= \{\,20.0 + (T_8 - 286.0) \times 0.41\,\} \times 10^{-3}\end{aligned}\right\}\ \text{for } 286 < T(k) < 305$$

$(k = 4, 6, 8)$

where, T_4, T_6 and T_8 denote the temperature (in Kelvin scale) at 400 mb, 600 mb and 800 mb respectively.

APPENDIX III

Convergence criteria for diabatic omega equation

Recalling (3·1·7), we had,

$$S^* - S < 0$$

as the necessary condition for the convergence of the omega equation with diabatic forcing. If FD denotes the forcing due to vorticity advection and the Laplacian of thermal advection and $\omega^{(1)}$ is the corresponding vertical velocity, following (2·8) we have,

$$\nabla^2 \omega^1 + \frac{f_0^2}{S}\frac{\partial^2 \omega^1}{\partial p^2} = FD \qquad \text{(A.III.1)}$$

Similarly following (3·1·6), we have,

$$\nabla^2 \omega^2 \pm \frac{f_0^2}{S}\frac{\partial^2 \omega^2}{\partial p^2} = FD + \nabla^2\left(\frac{S^*}{S}\omega^1\right) \qquad \text{(A.III.2)}$$

(where, ω^1, ω^2, ω^3 etc refer to omega with different forcing).

Subtracting (A.III.1) from (A.III.2), we have,

$$\nabla^2(\omega^2 - \omega^1) + \frac{f_0^2}{S}\frac{\partial^2}{\partial p^2}(\omega^2 - \omega^1) = \nabla^2\left(\frac{S^*}{S}\omega^1\right)$$

Therefore, approximating the above equations, we have,

$$\omega^2 - \omega^1 \approx \frac{S^*}{S}\omega^1$$

$$\therefore\ \omega^2 = \left(1 + \frac{S^*}{S}\right)\omega^1 \qquad \text{(A.III.3)}$$

Similarly, $\omega^3 - \omega^2 = \frac{S^*}{S}\omega^2$

Substituting (A.III.3), we have,

$$\omega^3 - \omega^2 = \frac{S^*}{S}\left(1 + \frac{S^*}{S}\right)\omega^1 = \left\{\frac{S^*}{S} + \left(\frac{S^*}{S}\right)^2\right\}\omega^1$$

Therefore, in general we have,

$$\omega^n - \omega^{n-1} = \left\{\frac{S^*}{S} + \left(\frac{S^*}{S}\right)^2 + \dots + \left(\frac{S^*}{S}\right)^{n-1}\right\}\omega^1$$

Therefore, it may be inferred that in order that solution may converge, *i.e.*,

$$\omega^{(n)} - \omega^{(n-1)} < E$$

where, E is some prespecified tolerance limit, it is necessary to satisfy the condition given in (3.1.7). In actual numerical computation, whenever this criteria is not satisfied, it is artificially imposed in order to make the solution converge.