

Steady non-Darcy seepage through confined aquifer of variable thickness

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ABSTRACT. A steady state analytical solution for the case of confined aquifer of variable thickness incorporating Forchheimer's non-linear velocity gradient response ($i=av+bv^2$) is presented. The solution is compared with the available solution for Darcian linear flow. The effect of non-linearity in the flow response on the discharge characteristics and the piezometric pressure distribution in relation to the corresponding linear case is brought out.

1. Introduction

For many decades, Darcy's simple linear flow law connecting velocity and gradient served an unique rôle in the field of flow through porous media. While using Darcy's linearity law for various field problems it is always cautioned about its applicability in the high velocity zone where inertial forces are comparable over viscous forces and thus causing a less than proportional increase of velocity with gradient. The deviations from Darcian linear response are expected when Reynolds number of flow exceeds unity (Taylor 1948). Various investigators have proposed various forms of velocity gradient relationships which are mainly based on extensive experimental results. A summary of published relations is given in Table 1. The relative merits and demerits of these relationships are critically examined by Dudgeon (1966).

Of late various field situations have been recognised where it is felt that for accurate predictions a suitable non-linear velocity gradient response will have to be employed. Typical examples of such situations are:

- (i) Flow through coarse grained soils and rockfill banks and dams.
- (ii) Flow in coarse grained aquifer under high drawdown especially in the area adjacent to pumping well.
- (iii) Flow in filtrates.

In recent past, several attempts have been made to analytically solve some of the steady and unsteady state field problems incorporating

one or the other form of non-linear velocity gradient relationship shown in Table 1. Amongst the steady state problems, Slepicka (1961) studied the discharge from wells placed in unconfined and confined aquifers of constant thickness. Volker (1969, 1975) studied the non-linear seepage in isotropic and anisotropic porous media, Valsangkar *et al.* (1975) studied the non-linear seepage in tranches. Basak (1976a, 1976b) reported analytical solutions for non-Darcy seepage in non-penetrating wells and through embankments. Existing solutions for unsteady state problem with non-linear flow response number very few. Hansbo (1960) and Schmidt and Westmann (1973) gave solutions for one dimensional unsteady flow through compressible porous media incorporating velocity gradient response of the type $v = Mi^n$. Basak (1975) solved the same problem with different initial condition. Basak (1975) also presented an analytical solution for unsteady flow in a semi-infinite embankment with $v = Mi^n$.

Out of the various types of non-linear velocity gradient responses as shown in Table 1, the two most widely used are that the due to Izbash (1931) and Forchheimer (1901). While Izbash's equation is purely empirical, Forchheimer's equation, though initially proposed on the basis of experimental results alone, is found to have theoretical justification also (Irmay 1958 and Ahmed 1967).

In this paper, a steady state analytical solution for the case of confined aquifer of variable thickness incorporating Forchheimer's flow ($i=av+bv^2$) law is presented. The effect of non-linearity in the flow response on the discharge

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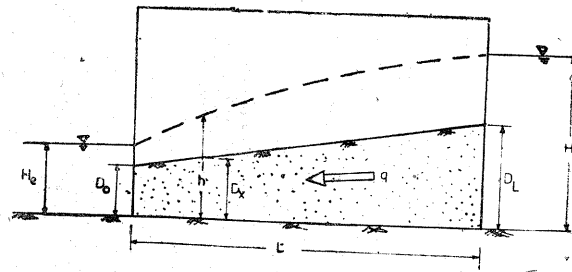


Fig. 1. Definition sketch

characteristics and piezometric distribution in relation to corresponding linear case with constant aquifer thickness (Harr 1962), is brought out.

2. Analysis

Fig. 1 gives the definition sketch for the problem and describes a uniformly sloping confined aquifer having thickness D_0 on the downstream side and D_L on the upstream side. At steady state, downstream water level is H_e and that of upstream is H . The aquifer has length L and it is resting on an impervious horizontal base. The steady state solution for the problem with constant aquifer thickness (i.e., $D_L/D_0 = 1$) obeying Darcian linear flow equation ($v = ki$ or $i = av$) is available (Harr 1962).

Choosing the origin on the downstream too as shown in Fig. 1, the macroscopic seepage velocity or superficial velocity, v at any distance x will be given by:

$$v = q/D_x \quad (1)$$

in which q is the discharge and D_x is the thickness of the aquifer at distance x from the downstream side and is given by:

$$D_x = D_0 + \left[\frac{D_L - D_0}{L} \right] x \quad (2)$$

As mentioned earlier, the velocity gradient response due to Forchheimer is:

$$i = -\frac{dh}{dx} = av + bv^2 \quad (3)$$

where h is the piezometric head at any distance x .

Combining Eqns. (1), (2) and (3) one obtains the governing differential equation as:

$$\frac{dh}{dx} = a \left[\frac{q}{D_0 + (D_L - D_0)x/L} \right] + b \left[\frac{q}{D_0 + (D_L - D_0)x/L} \right]^2 \quad (4)$$

This equation can be non-dimensionalised and rearranged as:

$$\frac{H}{L} \frac{dy}{dx} = \frac{q_*}{1 + \left(\frac{D_L}{D_0} - 1 \right)} \left[1 + \frac{cq_*}{1 + \left(\frac{D_L}{D_0} - 1 \right)} \right] \quad (5)$$

where,

$$y = \frac{h}{H} \quad (6)$$

$$x = \frac{X}{L} \quad (7)$$

$$q_* = \frac{a_0}{D_0} \quad (8)$$

$$c = b/a^2 \quad (9)$$

Eqn. (5) is the governing differential equation of the problem in non-dimensional form.

The coefficient c as defined in Eqn. (9), may be termed as non-Darcy index or parameter, when this index, i.e., $c=0$, the flow is perfectly linear and hence Darcian. An estimation of the non-Darcy parameter, c can be obtained from published values of the coefficients a and b for sands and gravel and the calculation of c therefrom. For sand, the value of the non-Darcy parameter c lies between 0.01 and 1.0 and that for gravel, it is generally found to be more than 2.

The boundary conditions of the problem are:

$$x = 0, h = H_e \text{ or } X = 0, y = \frac{H_e}{H} = y_0 \quad (10)$$

and

$$x = L, h = H \text{ or } X = 1, y = \frac{H}{H} = 1 \quad (11)$$

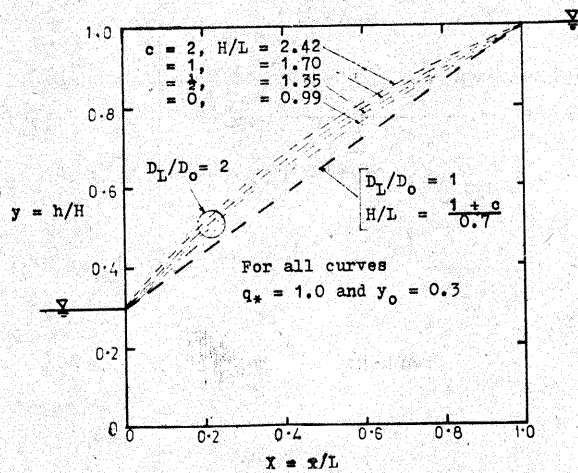


Fig. 2. Typical phreatic surfaces for various D_L/D_0 and non-Darcy parameter, c

Integration of Eqn. (5) along with the boundary condition (10) yields the equation for piezometric surface and is given by :

$$y = y_0 + \frac{q_*}{\left(\frac{H}{L} \frac{D_L}{D_0} - 1\right)} \left[\log \left\{ 1 + \left(\frac{D_L}{D_0} - 1 \right) X \right\} + cq_*^2 \left\{ 1 - \frac{1}{\left(\frac{D_L}{D_0} - 1 \right) X} \right\} \right] \quad (12)$$

When the flow is Darcian, the equation for piezometric surface for confined aquifer of variable thickness can be obtained by simply putting $c=0$ in Eqn. (12) and which reads as :

$$y = y_0 + \frac{q_*}{H/L} \frac{\left[\log \left(1 + \frac{D_L}{D_0} - 1 \right) X \right]}{\left(\frac{D_L}{D_0} - 1 \right)} \quad (13)$$

and for aquifer of constant thickness, Eqn. (12) reduces to (after taking limit as $D_L/D_0 \rightarrow 1$ and using La' Hospital's theorem):

$$y = y_0 + \left[\frac{q_* + cq_*}{H/L} \right] X \quad (14)$$

For Darcian flow ($c=0$), Eqn. (14) leads to the standard linear variation of piezometric head (Harr 1962) and is given by equation:

$$y = y_0 + \left[\frac{q_*}{H/L} \right] X \quad (15)$$

This is to be noted here that as long as the aquifer is of constant thickness, the piezometric head variation is linear irrespective of the nature of flow, whether Darcian or non-Darcian.

Typical piezometric surface (Eqns. 12 and 13) for non-Darcy parameter, $c=0.5, 1.0, 1.5$ and 2.0 for geometry parameter $D_L/D_0 = 2$ with $y_0=0.3$ and non-dimensional discharge $q_* = 1.0$ are drawn and shown in Fig. 2. This figure also includes the linear piezometric surface for constant aquifer thickness (i.e., $D_L/D_0 = 1$) with other parameters remaining same.

The expression for non-dimensional discharge q_* is obtained by inserting second boundary condition (Eqn. 11) in Eqn. (12) and is given by :

$$\frac{H}{L}(1-y_0) = q_* \left[\frac{\log D_L/D_0}{(D/L)/D_0 - 1} \right] + cq_*^2 \left[\frac{1-D_0/D_L}{D_L/D_0 - 1} \right] \quad (16)$$

which can be rewritten in the form similar to Forchheimer's equation itself and is given by :

$$I_{av} = Aq_* + Bq_*^2 \quad (17)$$

where,

$$I_{av} = \frac{H}{L}(1-y_0) = \text{Average gradient} \quad (18)$$

$$A = \left[\frac{\log D_L/D_0}{D_L/D_0 - 1} \right] \quad (19)$$

and

$$B = c \left[\frac{1-D_0/D_L}{D_L/D_0 - 1} \right] \quad (20)$$

For constant aquifer thickness, taking limit as $D_L/D_0 \rightarrow 1$, and using La'Hospital's rule values for both A and B become unity. For Darcian flow B is always zero (as $c=0$) and

TABLE 1

Published non-linear flow equations at high Reynolds numbers

Equation	Original proposer	Explanation of terms	Comments
(1) $i=av+bv^2$	Forchheimer (1901)	a, b =Constants with units of T/L and $(T/L)^2$ respectively	Empirical but theoretical basis was found later by Irmay (1958) & Ahmed (1967)
(2) $i=av+bv^2+cv^3$	Forchheimer (1901)	a, b, c =Constants	Empirical
(3) $i=av+bv^{1.5}+cv^2$	Rose (1951)	a, b, c =Constants	Empirical
(4) $i=av+bv^2+c(\partial x/\partial t)$	Poluberinova Kochina (1962)	a, b, c =Constants	Empirical
(5) $i=av+bvm$	Muskat (1946) and Harr (1962)	a, b and m =Constants	Empirical
(6) $v=Mi^n, n<1$	Izbash (1931)	m =Constant with unit of L/T n =Non-Darcy exponent	Empirical
(7) $i=avm, m<1$	Missbach (1937)	a =Constant n =Non-Darcy exponent	Empirical
(8) $v=(Bi)^{\frac{1}{2}}$	Escande (1953)	B =Constant	Empirical B varies between 80 & 290 (cm/sec) ² for particles of dia. 2.54 cm
(9) $v=32.9 m^{\frac{1}{2}} i^{0.54}$	Wilkinson (1956)	m =Hydraulic radius	Semi-empirical based on test results with particles of 1.905 cm to 7.62 cm
(10) $v = \left(\frac{\mu}{\sigma} \right)^{f-1} (ki)^f$ $f < 1$	Spepicka (1961)	f, k =Constants μ =Viscosity σ =Surface tension	Semi-empirical, derived from dimensional analysis

Eqn. (17) reduces to the Darcian discharge equation for confined aquifer of variable thickness and read as :

$$I_{av} = \left[\frac{\log(D_L/D_0)}{D_L/D_0 - 1} \right] q_* \tag{21}$$

and if the aquifer is of constant thickness (i.e., $D_L/D_0 \rightarrow 1$), then applying La'Hospitals rule, A becomes equal to unity and the average gradient versus non-dimensional discharge relationship reduces to the known (Harr 1962) simple form :

$$I_{av} = q_* \tag{22}$$

The non-dimensional discharge q_* as given by Eqn. (16) or (17) are plotted for various values of geometry parameter D_L/D_0 and non-Darcy parameter c against average gradient in Fig. 3. The error in discharge estimation by assuming

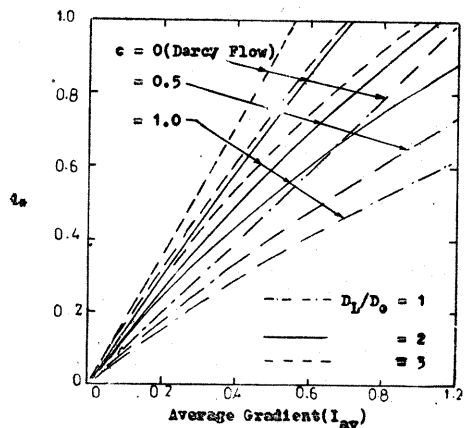


Fig. 3. Variation of non-dimensional discharge q_* with average gradient (I_{av}) for different geometry parameter (D_L/D_0) and non Darcy parameter (c)

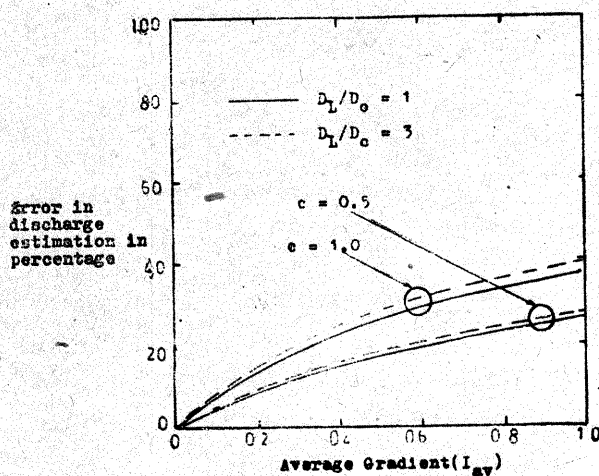


Fig. 4. Error involved in discharge estimation in assuming the flow to be Darcian as a function of non-Darcy parameter, c and the geometry parameter, D_L/D_0 .

the flow to be Darcian is plotted against the average gradient in Fig. 4.

3. Results and discussions

The piezometric head distribution as given by Eqn. (12) and Fig. 2 indicates that to maintain the same discharge, head at any distance from the downstream side is required to be more in case of non-Darcy flow than that of Darcian case. Higher is the non-Darcy parameter c , higher will be the piezometric head.

Eqns. (12), (13) & (14) and Fig. 2 also indicate that for all values of $D_L/D_0 \neq 1$, the piezometric head variation is non-linear and when thickness of the aquifer is constant (i.e., $D_L/D_0 = 1$), the piezometric head variation is always linear irrespective of the nature of flow (whether Darcian or non-Darcian).

It is evident from Eqns. (16) to (21) and Fig. 3 that the non-dimensional discharge q_* varies parabolically with the average gradient and the variations are very sensitive to the non-Darcy parameter, c for all value of D_L/D_0 ratios. The errors seem to be insensitive to geometry parameter, D_L/D_0 . Exact values of error involved can be read from Fig. 4.

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