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A probabilistic model for estimating ocean-spray distribution in extreme tropical cyclone winds

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सार — 28 मी./सेकेंड से भी अधिक तीव्र वायु गति से आने वाली समुद्री फुहार के संबंध में जानकारी की भारी कमी को प्रेक्षण अथवा प्रयोग से पूरा कर पाना कठिन कार्य है। फिर भी उष्णकटिबंधीय चक्रवातों (टाइफून और झंझावात सहित) के तापगतिकी के सूक्ष्म अध्ययन के लिए इस परिकल्पना के मूल्यांकन की आवश्यकता है कि वायु की अधिकतम तीव्र गति से आने वाली समुद्री फुहारों की आर्द्रता से (*i*) आईवॉल में उठने वाली संतृप्त वायु का आरम्भिक तापमान और (*ii*) समूचे वायुप्रवाह में नैसर्गिक ऊर्जा के आदान में तीव्र गति से होने वाली इस तरह की प्रगामी कमी के प्रभाव भविष्य में देखने को मिल सकते है। उदाहरणार्थ, उष्णकटिबंधीय चक्रवातों के व्यापक रूप से स्वतः परिसीमित प्रचंडीकरण पर भविष्य में विश्व स्तरीय उष्णन का कदाचित कोई प्रभाव पड़ सकता है। वायु के अधिकतम वेग से आने वाली समुद्री फुहारों के विषय में जानकारी के वर्हिगणन में सहायता के लिए, एक अनुमान पर आधारित विश्लेषण का सूत्रपात किया गया है, जिससे वायु झोंकों, उसकी तीव्रता और वाष्पीकरण के जलकणों के वितरण पर पड़ने वाले प्रभाव का पता चलता है, तथापि अन्य मामलों के लिए भी यह विश्लेषण पद्धति बहुत ही सरल है। इस सरल विश्लेषण से प्राप्त आरम्भिक संकेत समुद्री फुहारों की आर्द्रता के सम्भाव्य महत्व की पुष्टि करते है।

ABSTRACT. Serious gaps in knowledge about ocean spray at wind speeds over 28 m/s remain difficult to fill by observation or experiment; yet refined study of the thermodynamics of Tropical Cyclones (including typhoons and hurricanes) requires assessment of the hypothesis that 'spray cooling' at extreme wind speeds may act to reduce (*i*) the initial temperature of saturated air rising in the eyewall and so also (*ii*) the input of mechanical energy into the airflow as a whole. Such progressive reductions at higher speeds could, for example, make any possible influence of future global warming on Tropical Cyclone intensification largely self-limiting. In order to help in extrapolation of knowledge on ocean spray to extreme wind speeds, a probabilistic analysis is introduced which allows for the effects of gusts, gravity and evaporation on droplet distributions yet in other respects is as simple as possible. Preliminary indications from this simplified analysis appear to confirm the potential importance of spray cooling.

Key words — Boundary layer, Droplets, Evaporation, Global warming, Gusts, Probabilistic model, Spray, Spray cooling, Tropical cyclones, White caps.

1. Introduction

Input of mechanical energy into tropical cyclone airflows takes place especially in the eyewall, by the action of buoyancy forces on air that has reached saturation during its long cyclonically spiralling path over the ocean. Provided that the ambient atmosphere has its temperature drop with height significantly steeper than that of rising saturated air (*i.e.* 'lapse rate exceeds the moist-air adiabatic'), then such air finds itself

everywhere warmer than its surroundings and its further ascent is powered by buoyancy forces.

On the other hand, because vapour pressure increases sharply with temperature, this input of mechanical energy (essential, of course, to balance turbulent dissipation in the atmospheric boundary layer) is itself a steeply increasing function of the temperature T_1 at which saturated air begins its rise in the eyewall. It is this functional dependence that suggests both (*i*) why it is only over tropical oceans that tropical cyclones form, and also (*ii*) why, on the other hand, 'spray cooling' (a tendency for effects of ocean spray, as winds become more intense, to reduce T_1 more and more below the sea surface temperature) may set a limit on any increase in tropical cyclone intensities with ocean temperature (Lighthill 1997).

At high wind speeds there is, indeed, between ocean and atmosphere, a thick layer of 'a third fluid': ocean spray, consisting of a relatively tall cloud of droplets. Many of the smaller ones (with radii not more than about 20 μm) appear when air bubbles burst at the sea surface. A greater mass of droplets, however, is formed (Andreas *et al.* 1995) either as 'splash' torn from, or as 'spume' ejected from, whitecaps (in the form of droplets with radii ranging from 20 μm to much larger values). A recent survey (Fairall *et al.* 1994) estimated that, at a wind speed of 40 m/s (the speed at which the fraction of surface covered by whitecaps is expected to reach unity), spray droplets have attained a volume concentration of about 10^{-5}, corresponding to a mass concentration of about 10^{-2}, and yet that vapour transfer from droplets to wind exceeds by an order of magnitude any direct transfer of vapour from the ocean surface.

The well organised project HEXOS (Humidity Exchange over the Sea) made (Smith *et al.* 1996) admirably systematic vapour- transfer and heat-transfer measurements - especially at Meetpost Noordwijk - in a wide range of North Sea weather conditions up to a maximum wind speed of 18 m/s. By contrast, Russian research ships have made measurements at wind speeds up to 28 m/s in the general vicinity of Pacific Ocean typhoons. These Russian measurements (Pudov 1993), while supporting HEXOS data in finding little effect of spray in winds up to 18 m/s, recorded a rather large increase of spray mass as wind speeds rose to 28 m/s, alongside an increasing level of 'spray cooling' depressing wind temperatures below sea surface temperatures (Fairall *et al.* 1994).

On the other hand, the extreme wind speeds found in tropical cyclones line in the range 50 to 60 m/s. For proper discussion of tropical cyclone thermodynamics and related topics, therefore, it may be necessary to attempt an assessment of ocean spray and its effects (including possible 'spray cooling') at such wind speeds around twice those for which any measurements have yet been made. It is principally with the aim of facilitating such an extrapolation that a simplified model of the fluid dynamics of ocean spray is introduced in this paper. At the same time, to counter possible charges that the significance of spray is being exaggerated, relatively 'cautious' choices of simplifying assumptions are made (that is, those which may tend - if anything - to underestimate heights of spray clouds).

2. A simplified spray model

The three physical effects that compete to influence the vertical distribution of those spray droplets which are emitted from the ocean surface are (*i*) gusts, (*ii*) gravity and (*iii*) evaporation. For the present limited aim of extrapolation to wind speeds around twice those for which spray measurements have been made, a radically simplified spray model may suffice provided that all three of these key effects are taken into account. Such a model is here outlined in section 2, and then elaborated in more detail in sections 3, 4 and 5 with allowance made respectively for effect (*i*) alone, for effects (*i*) and (*ii*) together, and finally for the simultaneous action of all three effects.

In gusts, it is above all the vertical component, w, of air velocity whose statistical properties influence (alongwith the effects of gravity and evaporation) how emitted droplets become vertically distributed. To a crude approximation, we may think of a parcel of air being subject to a random succession of gusts creating different vertical movements (up or down), while a droplet within the parcel falls relatively to it at roughly its terminal velocity; and, also, while that terminal velocity is gradually retarded as the droplet radius becomes diminished by evaporation. This crude picture underlies the simplified spray model that is introduced below.

Of course Taylor (1921), in his celebrated paper 'Diffusion by continuous movements', rightly emphasized how diffusion in a continuous, albeit turbulent,

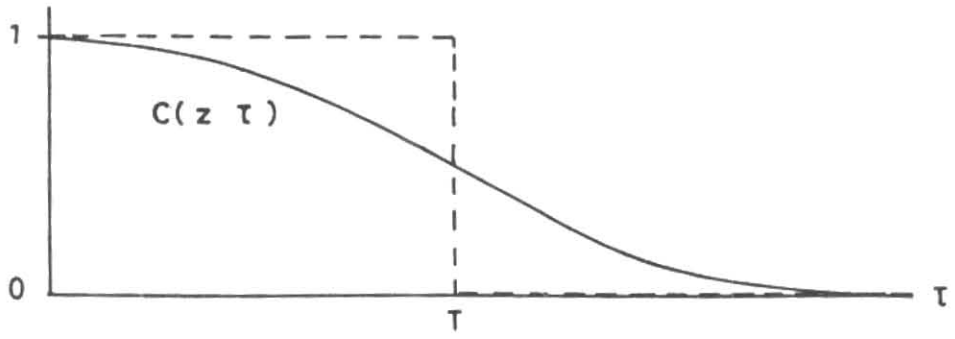

Fig. 1. Solid line: typical variation of $C(z, \tau)$ with τ. Broken line (discontinuous): simplifying approximation

fluid flow differs essentially from diffusion associated with those random movements of molecules which undergo discontinuities wherever two molecules collide. For mathematically describing the effects of continuously varying random movements of a fluid particle, Taylor introduced correlation functions of the type which nowadays are generally called 'Lagrangian' (to distinguish them from the Eulerian correlation functions which later came to be used still more widely); more recently, attention was drawn by Hunt (1985) to the persistent value of using such functions in diffusion studies.

In an atmospheric boundary layer with statistical properties which are horizontally homogeneous, the appropriate Lagrangian correlation function for characterizing random vertical displacements of a particle (small parcel) of air may be written

$$C(z, t) = \frac{\langle w(t)\, w(t+\tau) \rangle}{\langle [w(t)]^2 \rangle} \qquad (1)$$

Here, $w(t)$ represents the particles vertical component of velocity at a certain time t when it is at height z, while $w(t+\tau)$ represents, for any $\tau > 0$, the vertical component of velocity for the same particle at a later time $t+\tau$ (when in general it is at a different height); angle brackets signify an average over all such particles.

The definition Eqn. (1) makes $C \to 1$ as $\tau \to 0$, while the random nature of turbulence ensures that C is effectively zero whenever τ is large. Furthermore the integral

$$T(z) = \int_0^\infty C(z, \tau)\, d\tau \qquad (2)$$

which has the dimensions of time, is often called the Lagrangian correlation time. Essentially (see Fig. 1), it is a measure of that time-difference within which values of C remain significant; to a crude approximation, the correlation C (z, τ) is substantial when $\tau < T$ and yet relatively insignificant when $\tau > T$. Thus T is a sort of 'time of flight' for the coherent vertical displacement of a small parcel of air, after which the immediately succeeding vertical displacement can almost be considered as if it were statistically independent. Such a view of the Lagrangian correlation time T (z) may at least be more valuable than were similar views (once fashionable) about 'mixing lengths' - above all, because time is one-dimensional; whereas the three-dimensional character of space, and moreover the constraints on velocity fields provided by the equation of continuity, place obstacles in the way of any spatial analogue to the simplifying interpretation given in Fig. 1.

The present model not only adopts this crude simplification (assuming statistically independent vertical displacements of a parcel of air in successive times of flight) but also — with the principal object of maximum simplicity - uses for the time of flight a single uniform value T independent of z. That additional simplification is made even though, in established descriptions of turbulent boundary layers over rough solid surfaces, the Lagrangian correlation time T increases with height z. Admittedly, the atmospheric boundary layer over a deeply heaving ocean surface might, for small heights z above mean sea level, involve increased coherence of vertical motion, which would tend to smooth out the variation

of T with z; even so, the primary purpose of assuming a uniform time of flight T is to obtain in spray model simple enough to facilitate extrapolation to greatly increased wind speeds. The assumption may be 'cautious' (in the sense suggested at the end of section 1) because it neglects enhanced vertical displacements experienced by droplets that have reached high levels.

Once the time of flight T has been taken independent of z, it is logical to postulate the same height-independence for a key probability distribution $g(\zeta)$ which will be called 'the gust function'. Here $g(\zeta)$ is the probability distribution for ζ, the vertical displacement of a parcel of air in the fixed time T. (Its height independence can reasonably be assumed from the fact that measured root-mean-square values for the vertical component of velocity are practically uniform across a turbulet boundary layer.) If R is the range of possible values of ζ, then the gust function $g(\zeta)$ satisfies the equations

$$\int_R g(\zeta)\, d\zeta = 1, \int_R \zeta g(\zeta)\, d\zeta = 0 \ , \int_R \zeta^2 g(\zeta)\, d\zeta = G \tag{3}$$

for a probability distribution with zero mean and with variance G. In what follows, $g(\zeta)$ will normally be taken as an even function with range $|\zeta| < Z$.

This paper's primary concern is to estimate $f(z, t)$, the probability distribution for the height z of a spray droplet at time t after leaving the level $z = 0$. During the time of flight T a droplet within a parcel of air descends relatively to it by a distance close to VT where V is its terminal velocity (at least, on a 'cautious' assumption that ignores any possible levitating influence of such coherent eddy motions as may help to keep droplets aloft in rain clouds). This implies that (z, t) satisfies the integro-difference equation

$$f(z - VT, t + T) = \int_R f(z - \zeta, t)\, g(\zeta)\, d\zeta; \tag{4}$$

where the right-hand convolution of the probability distribution for the droplet being at height $(z - \zeta)$ at time t with the probability of the parcel air being displaced by ζ, during the time of flight T gives the probability distribution for the parcel being at height z, and so also for the droplet being at height $z - VT$, at time $(t + T)$.

Evidently, the initial condition appropriate to solutions of this integro-difference equation is

$$f(z, 0) = \delta(z) \tag{5}$$

because the probability distribution $f(z, t)$ by its definition is concentrated at just one value $z = 0$ at time $t = 0$. The boundary condition, on the other hand, needs more careful consideration.

Essentially, this boundary condition must take into account the fact that the life of a spray droplet cannot continue after it has once returned to the ocean surface. Admittedly, that surface's height changes continually; here, however, the cautious assumption is made that a droplet disappears as soon as it has regained its initial height; in other words, when z becomes zero. This assumption may be described as cautious (erring on the side of over-predicting the reabsorption of spray droplets) simply because the majority of droplets are believed to be generated at levels higher than mean ocean-surface levels. For applying such a boundary condition to the integro-difference Eqn. (4), it is sufficient to specify (as an overriding requirement) that

$$f(z, t) = 0 \qquad \text{for all } z < 0 \tag{6}$$

This excludes from the range of integration R all values of ζ greater than z; while, still more simply, it requires Eqn. (4) to be ignored whenever $z < VT$.

In the next two sections, exact solutions of Eqn. (4) under conditions in Eqns. (5) and (6) are compared with exact solutions of a partial differential derived from it by an approximation scheme of Fokker-Planck type. In this scheme, the expression $f(z - \zeta, t)$ is approximated as just the first 3 terms of its Taylor series, to make the right- hand side

$$\int_R \left[f(z, t) - \zeta \frac{\partial f}{\partial z} + \frac{1}{2} \zeta^2 \frac{\partial^2 f}{\partial z^2} \right] g(\zeta)\, d\zeta, \tag{7}$$

which by Eqn. (3) is

$$f(z, t) + \frac{1}{2} G \frac{\partial^2 f}{\partial z^2} \ . \tag{8}$$

The left-hand side is then approximated as

$$f(z, t) + T \left(\frac{\partial f}{\partial t} - V \frac{\partial f}{\partial z} \right) \tag{9}$$

to yield a partial differential equation of convection-diffusion type,

$$\frac{\partial f}{\partial t} - V\frac{\partial f}{\partial z} = D\frac{\partial^2 f}{\partial z^2}\ , \quad \text{where } D = \frac{G}{2T} \tag{10}$$

is the diffusivity. In the 'comparison' sections 3 and 4, appropriate solutions of Eqn. (10) are found to represent solutions of Eqn. (4) in an asymptotic sense, with quite reasonable accuracy achieved already for surprisingly modest values of t/T.

3. Comparisons in the 'weightless drops' case

Such comparisons are attempted first in the case $V = 0$: the 'weightless drops' case with gravity neglected. Then Eqn. (10) becomes the pure diffusion equation,

$$\frac{\partial f}{\partial z} = D\frac{\partial^2 f}{\partial z^2}, \text{ again with } D = \frac{G}{2T} \tag{11}$$

A boundary condition appropriate to Eqn. (11), corresponding to the exact boundary condition that droplets disappear on once again reaching $z = 0$, can be obtained by writing in two ways the diffusive transport of droplets into $z = 0$ during time T as

$$T\left[\ D\left(\frac{\partial f}{\partial z}\right)_{z=0}\ \right] = \int_0^z f(z, t)\,\mathrm{d}z \int_z^Z g(\zeta)\,\mathrm{d}\zeta \tag{12}$$

Here the square-bracketed diffusive flux into the surface $z = 0$ is multiplied by T to give the transport per unit area in time T; which also, can be written as a convolution of the probability distribution $f(z\ ,\ t)$ for a droplet being at height z with the probability

$$\int_{-Z}^{-z} g(\zeta)\,\mathrm{d}\zeta \tag{13}$$

for a droplet making a vertical displacement of $-z$ or less during time T. Here, $|\zeta| < Z$ is the range of $g(\zeta)$, and the integral in Eqn. (13) can be rewritten as an integral from z to Z because $g(\zeta)$ is an even function.

Now Eqn. (12), with reversed order of integration and $f\ (z\ ,\ t)$ represented by two terms of a Taylor series, while D is substituted from Eqn. (11), becomes

$$\frac{1}{2}G\left(\frac{\partial f}{\partial z}\right)_{z=0} \doteq \int_0^z g(\zeta)\,\mathrm{d}\zeta \int_0^\zeta \left[\ f(0, t) + z\left(\frac{\partial f}{\partial z}\right)_{z=0}\right]\mathrm{d}z$$

$$= Jf(0, t) + \frac{1}{4}G\left(\frac{\partial f}{\partial z}\right)_{z=0}, \quad \text{where } J = \int_0^Z \zeta g(\zeta)\,\mathrm{d}\zeta \tag{14}$$

is the one-sided moment of the distribution $g(\zeta)$. Eqn. (14) gives the required boundary condition in the relatively simple form,

$$f = A\frac{\partial f}{\partial z} \text{ on } z = 0\ , \quad \text{with } A = \frac{G}{4J} \tag{15}$$

In this approximate boundary condition in Eqn. (15), the quantity A has the dimensions of length and bears a simple relationship to the root-mean-square deviation $\sigma = G^{1/2}$ of the probability distribution $g(\zeta)$. Thus

$$\frac{A}{\sigma} = \frac{\sigma}{4J} \tag{16}$$

which for all possible distributions has the rigorous minimum 0.500 while taking values only a little greater than 0.500 for familiar forms of $g(\zeta)$. For example, it is 0.577 for the 'top-hat' distribution

$$g(\zeta) = \frac{1}{2Z}\ (|\zeta| < Z),\ \ 0\ (|\zeta| > Z) \tag{17}$$

with sharp discontinuities at $\zeta = \pm Z$; yet, for a perfectly smooth Gaussian distribution it is 0.627.

Actually, Eqn. (11) for f possesses Gaussian solutions, and a multiple of the first derivative of one of these is the solution

$$f = B\left(\frac{T}{\pi t}\right)^{1/2}\frac{z + A}{2Dt}\ e^{-\frac{(z+A)^2}{4Dt}} \tag{18}$$

which may be recognised as satisfying the boundary condition in Eqn. (15) asymptotically; that is, when t is large compared with

$$\frac{A^2}{2D} = \left(\frac{A}{\sigma}\right)^2 T \tag{19}$$

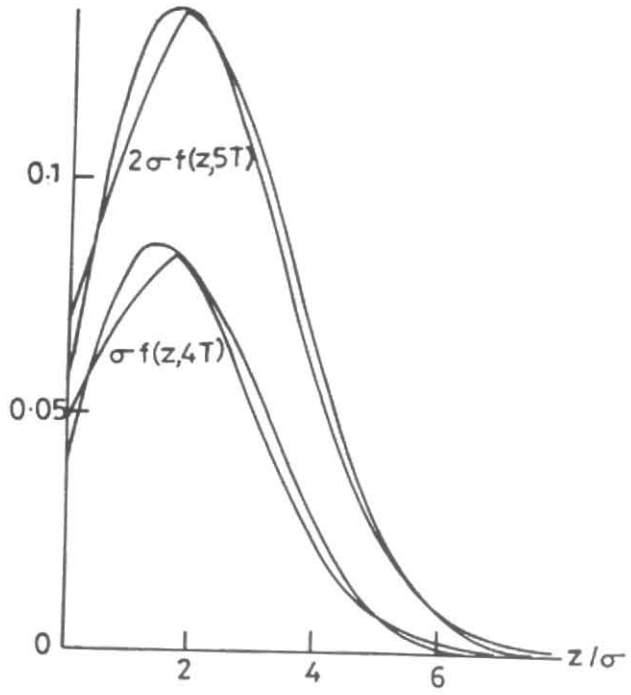

Fig. 2. Comparisons, for $V = 0$, between exact (kinked) and asymptotic smooth forms of $\sigma f(z, 4T)$ and $2\sigma f(z, 5T)$

With the ratio in Eqn. (16) taking values as small as those just mentioned, this condition could perhaps be met for values of t/T as low as 4 or 5.

Fig. 2 confirms that this solution in Eqn. (18) of Eqn. (11) which asymptotically satisfies the boundary condition in Eqn. (15) does indeed (with the choice $B = 1$ for the multiplying constant) compare closely for $t/T = 4$ or 5 with computed solutions of the integro-difference Eqn. (4) under the boundary condition in Eqn. (6). This is an exacting comparison, made for the case when $g(\zeta)$ has the 'top-hat' form in Eqn. (17) with its discontinuities at $|\zeta| = Z$. These lead to discontinuities of slope at $z = Z$ in $f(z, t)$, yet representation by the perfectly smooth curve of Eqn. (18) is a reasonably close one.

4. Analysis when V is a positive constant

A similar comparison is now made in the case when, besides gusts, gravity is taken into account (so that $V > 0$) but evaporation is as yet ignored (so that V = constant). The partial differential equation now takes the form of Eqn. (10), of which an appropriate

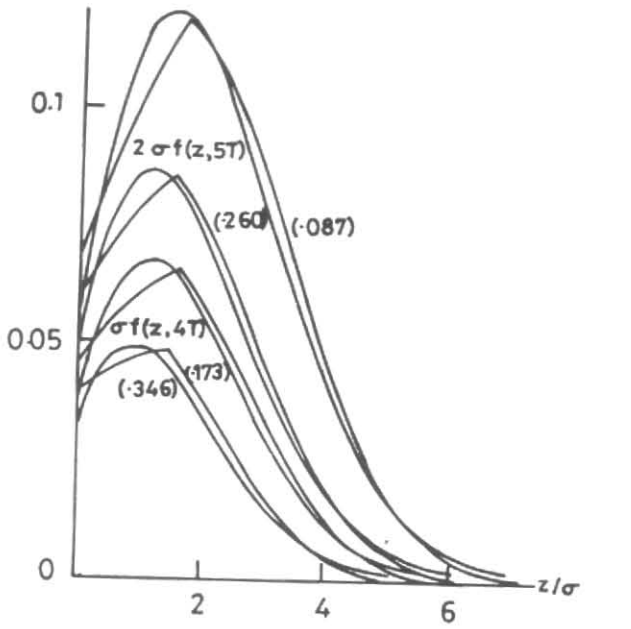

Fig. 3. Similar comparisons as in Fig. 2 for values of VT/σ given in brackets

exact solution is,

$$f = B\left(\frac{T}{\pi t}\right)^{1/2} \frac{z+A}{2DT}\, e^{-\frac{(z+Vt+A)^2}{4Dt}} \qquad (20)$$

This modified form of Eqn. (18), with z replaced by $z + Vt$ in the exponent (but not in the multiplying factor, so that condition given by Eqn. (15) is still closely satisfied) was suggested by numerical solutions of Eqn. (4), which all show portions farthest to the right for curves with $V > 0$ displaced to the left by about Vt from similar portions with $V = 0$; the far right being where variations in the exponential dominate over variations in the multiplying factor. (Of course, replacing z by $z + Vt$ in the solution Eqn. (18) of Eqn. (11) must give a solution of Eqn. (10) - of which, however, the part with Vt in the multiplying factor is just an exact Gaussian solution, leaving the difference given by Eqn. (20) as itself an exact solution).

Fig. 3 gives four illustrative comparisons of Eqn. (20), this time with the choice

$$B = 1 + \frac{VT}{\sigma} \qquad (21)$$

for multiplying constant, against computed solutions of the integro-difference Eqn. (4) for the top-hat form Eqn. (17) of $g(\zeta)$. Agreement continues to

appear close enough for Eqn. (20) to be viewed as a useful asymptotic form of f.

Now, before any attempt is made to include evaporation in the analysis, implications of this asymptotic form for steady-state volume distributions of droplets may be looked into, beginning with a brief study of the 'weightless drops' case ($V = 0$) of section 3. In this case, if S is the number of droplets generated per unit area per unit time, then $S\,dt$ multiplied by the height distribution $f(z, t)$ for droplets after time t gives the volume distribution of those droplets that were emitted at times from t to $(t + dt)$ earlier than the present time. Therefore, the volume distribution of all droplets (regardless of when they were emitted) is $SF(z)$ where

$$F(z) = \int_0^\infty f(z, t)\,dt \tag{22}$$

Now, for larger values of z (say, $z \geq 5\sigma$), exact solutions of the integro-difference equation vanish for smaller times (say $t \leq 3T$). Yet for larger t/T they are well represented by the asymptotic solution given by Eqn. (18), which therefore can be used in expression of Eqn. (22). This integral is readily evaluated by a substitution,

$$\tau = \frac{(z+A)^2}{4Dt},\ d\tau = -\frac{(z+A)^2}{4Dt^2}\,dt, \tag{23}$$

to give

$$F(z) = B\left(\frac{T}{\pi D}\right)^{1/2} \int_0^\infty \tau^{-1/2} e^{-\tau}\ d\tau$$

$$= B\left(\frac{T}{D}\right)^{1/2} = \frac{T}{\sigma} 2^{1/2} \tag{24}$$

a uniform volume distribution. In words, 'weightless droplets reach arbitrary heights'. (Note: those powerful fliers, the swifts, are frequently observed in strong winds feeding at heights exceeding 1 km on the aeroplankton of insects which remains abundant at such heights).

Conclusions, as might be expected, are very different when the terminal velocity takes for each droplet a constant value $V > 0$. Then a 'source function' $S(V)$ must be defined so that $S(V)dV$ is the rate of production per unit area per unit time of droplets with fall speeds between V and $V + dV$. Now it follows as before (*i*) that $[S(V)dV]F(z)$ is the volume distribution at height z for this group of droplets, and (*ii*) that, in the integral expression of Eqn. (23) for $F(z)$, the new asymptotic form given by of Eqn. (20) can be used for $f(z, t)$.

After the substitution given in Eqn. (23), the exponent within that asymptotic form given by Eqn. (20) is

$$\tau + \eta + \frac{\eta^2}{4\tau},\quad \text{where } \eta = \frac{V(z+A)}{2D}\ ; \tag{25}$$

so that the integral at Eqn. (22) becomes

$$F(z) = B\left(\frac{T}{\pi D}\right)^{1/2} e^{-\eta}\left[\int_0^\infty \tau^{-1/2}\, e^{-\left(\tau + \frac{\eta^2}{4\tau}\right)} dt\right] \tag{26}$$

The large brackets in Eqn. (26) enclose a standard integral from Bessel-function theory, equal to $\pi^{1/2} e^{-\eta}$; which, with the definition given by Eqn. (25) of η, yields

$$F(z) = B\left(\frac{T}{\pi D}\right)^{1/2} e^{-V(z+A)/D} = C\left(\frac{T}{D}\right)^{1/2} e^{-Vz/D}; \tag{27}$$

here, by Eqn. (21),

$$C = Be^{-VA/D} \doteq \left(1 + \frac{VT}{\sigma}\right) e^{-1.2\,VT/\sigma} \tag{28}$$

for a typical vlaue of A/σ around 0.6. Clearly, C is very close to 1 for those small fall speeds V which are of primary interest.

Essentially, then the volume distribution of all droplets at height z is close to

$$\left(\frac{T}{D}\right)^{1/2} \int_0^\infty S(V)\, e^{-Vz/D}\, dV \tag{29}$$

Thus, it is a simple multiple of the Laplace transform of $S(V)$.

There is, of course, nothing very surprising in the conclusion that a steady-state distribution given by Eqn. (27) may be proportional to $e^{-Vz/D}$; which, of course, is among the steady-state solutions of the convection-diffusion Eqn. (10). Yet no information regarding the amplitude factor outside that exponential would have emerged in any analysis using the steady-state form of Eqn. (10); while, even more crucially, such analysis would have been incapable of

taking evaporation into account – as can now be attempted from the present approach dependent on the development with time t of the probability distribution $f(z, t)$ of the height of an individual droplet.

5. Model allowing for evaporation

During this development with time, the droplet's terminal velocity V decreases as a result of evaporation. Indeed, V is a known function of the droplet's radius r; while, for given wind conditions (principally, relative humidity), evaporation reduces r at a rate which may depend both on r and V. It follows that the fall speed's rate of decrease can itself be written as a function of r and V and therefore (because of the relationship between them) as a function $E(V)$ of V itself.

$$\frac{dV}{dt} = -E(V) \tag{30}$$

For a droplet whose fall speed V decreases in this way after it leaves the surface when $t = 0$, its net downward motion during time t, relative to that parcel of air in which it is situated, becomes

$$X = \int_0^t V\,dt, \tag{31}$$

and it may appear plausible to use X in place of Vt in the formed asymptotic solution given by Eqn. (20), which would then become

$$f(z, t) = B\left(\frac{T}{\pi t}\right)^{1/2} \frac{z+A}{2Dt}\, e^{-\frac{(z+X+A)^2}{4Dt}} \tag{32}$$

As noted after Eqn. (20), this involves a simple shift (from z to $z + X$) within the exponent to allow for downward displacement of the droplet relative to an air particle, while interfering insignificantly with satisfaction of the boundary condition given in Eqn. (15).

In general, however, Eqn. (32) is no longer an exact solution of Eqn. (10); whose right-hand side it equates to,

$$B\left(\frac{T}{\pi t}\right)^{1/2} \frac{1}{2Dt}\, e^{-\frac{(z+X+A)^2}{4Dt}} \left[-\frac{3(z+A)}{2t} + \frac{(z+A)(z+X+A)^2}{4Dt^2} - \frac{X}{t}\right] \tag{33}$$

while its left-hand side takes the same form with $dX/dt = V$ replacing X/t as the last of the terms within large brackets. Thus they exactly coincide only when V is constant; on the other hand, the difference between them may plausibly be considered slight enough for Eqn. (32) to be regarded as a useful approximate solution of Eqn. (10). Then this expression may, for a drop where V takes the value V_0 when it leaves the surface at time $t = 0$, be used along with two integral relationships.

$$t = \int_V^{V_0} \frac{dV}{E(V)}, \quad X = \int_V^{V_0} \frac{V\,dV}{E(V)} \tag{34}$$

between t, X on the one hand and V_0, V on the other.

Earlier, the source function $S(V)$ was defined in such a way that $[S(V_0)\,dV_0]dt$ is the production per unit area in time interval dt of droplets with fall speeds between V_0 and $V_0 + dV_0$. In addition, a spray density function $s(z, V)$ can be defined so that $s(z, V)\,dV$ is the number per unit volume at height z of droplets in the fall-speed interval dV. These two functions are linked by the simple integral relationship

$$s(z, V) = \frac{1}{E(V)} \int_0^\infty S(V_0)\, f(z, t)\, dV_0 \tag{35}$$

in which of course $f(z, t)$ is a probability distribution per unit height for a spray droplet at time t after it leaves the surface. In Eqn. (35), multiplication of an area distribution $[S(V_0)dV_0]\,dt$ by a height distribution $f(z, t)$ yields after integration a volume distribution $s(z, V)\,dV$; and Eqn. (35) then follows because the factor outside the integral is the ratio $|dt/dV|$.

At larger heights (say, $z > 5\,\sigma$) the asymptotic form given by Eqn. (32) of $f(z, t)$ may be used in the relationship of Eqn. (35) between $s(z, V)$ and $S(V_0)$; provided that t and X, where they appear in it, are expressed in terms of V_0 and V by Eqns. (34). Then the integral given by Eqn. (35) lends itself to steepest-descents estimation because of the exponential factors in the form of Eqn. (32) for f.

As a function of V_0, the exponent,

$$\frac{(z+X+A)^2}{4Dt} \tag{36}$$

needs in a steepest-descents estimate to be minimised. But this exponent, with t and X given by Eqn. (34), has zero first derivative where

$$\frac{V_0}{E(V_0)}\frac{z+X+A}{2Dt}-\frac{1}{E(V_0)}\frac{(z+X+A)^2}{4Dt^2}=0 \qquad (37)$$

This gives a simple condition

$$z+X+A=2V_0t \qquad (38)$$

as the basic relationship identifying the predominant initial fall speed V_0 for those droplets which at height z, through evaporation, have acquired a given reduced fall speed V. The relationship given by Eqn. (38) appears to be a nontrivial conclusion from the line of argument presented in this paper.

At the value of V_0 for which the exponent's first derivative given by Eqn. (37) vanishes, its second derivative takes the positive value,

$$k=\frac{V_0}{E(V_0)}\left[1+\frac{V_0}{2t\,E(V_0)}\right]>0\ ; \qquad (39)$$

so the stationary point really is a minimum and steepest-descents estimation can be applied. It approximates the integral relationship of Eqn. (35) as

$$s(z,V)=\frac{1}{E(V)}S(V_0)f(z,t)\left(\frac{2\pi}{k}\right)^{1/2} \qquad (40)$$

with V_0 given in terms of V by Eqn. (38) while, of course, t and X are given by Eqn. (34).

Now this section ends by setting out numerical results for a specially simple case with the convenient property that all the conclusions can be displayed in a single diagram. It is the case when $E(V)$ takes just a constant value.

Actually, computations of $E(V)$ at pressures and temperatures typical of a Tropical Cyclone, and at relative humidity r_H, show [1] that $E(V)$, while decreasing, falls by less than a factor of 2 as V increases from 0 to 1m/s (droplet radius rising from 0 to 0.15 mm). Thus the constant-E approximation - while adopted here mainly as a conveniently concise way of illustrating the model is not impossibly unrealistic. (An appropriate value for the constant, with V in the above range, is 0.03 $(1 - r_H)$ m/s^2, from which $E(V)$ deviates by +30% at the lower end and by –30% at the upper.)

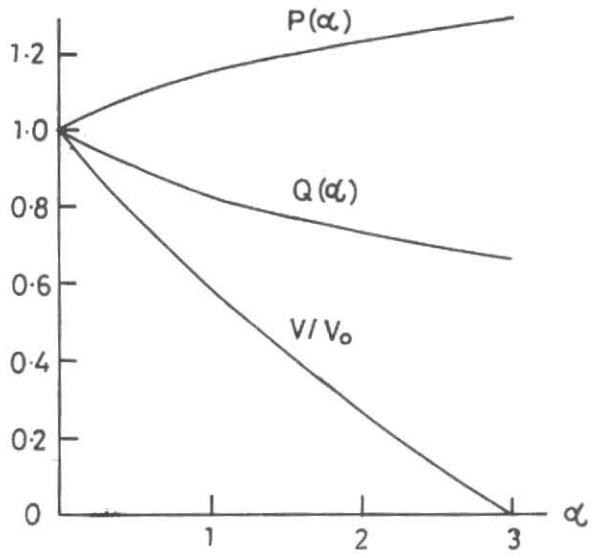

Fig. 4. Dependence of V/V_0, $P(\alpha)$ and $Q(\alpha)$ on α

For constant E, Eqns. (34) become $Et = V_0 - V$ and $2EX = V_0^2 - V^2$, so that the condition given in Eqn. (38) linking V_0 and V is a quadratic equation,

$$2E(z+A)=3V_0^2-4VV_0+V^2 \qquad (41)$$

and V/V_0 can be expressed, in terms of a non-dimensional variable,

$$2E(z+A)\,V_0^{-2}=\alpha,\ \text{as}\ V/V_0=2-(1+\alpha)^{1/2} \qquad (42)$$

Fig. 4 shows how V/V_0 is reduced as the measure of height α rises; note that, on the steepest-descents approximation, droplets with initial fall speed V_0 have evaporated completely (since $V = 0$) where $\alpha = 3$; that is, where

$$z+A=\frac{3V_0^2}{2E} \qquad (43)$$

Yet their number density has decayed with height somewhat less steeply than was suggested for cases without evaporation by Eqn. (27), simply because of the retardation in terminal velocity V as z increases.

Indeed the number density for all spray droplets at height z may be written as an integral

$$s_0(z) = \int_0^\infty s(z, V)\, dV \qquad (44)$$

with respect to V, which can be reformulated as an integral with respect to V_0 by use of the relationship of Eqn. (41) between V and V_0. With Eqn. (40) for $s(z, V)$, this gives,

$$s_0(z) = B\left(\frac{T}{D}\right)^{1/2} \int_0^\infty S(V_0)\, P(\alpha)\, e^{-Q(\alpha)\, V_0 (z+A)/D}\, dV_0 \qquad (45)$$

where the non-dimensional expressions,

$$Q(\alpha) = 2\alpha^{-1}\,[(1+\alpha)^{1/2} - 1],\ P(\alpha)$$
$$= \frac{1}{2}\left[\,1 + (1+\alpha)^{-1/2}\,\right]\left[\,2(1+\alpha)^{1/2} - 1\,\right]^{1/2} \qquad (46)$$

are also plotted in Fig. 4. With no evaporation, $\alpha = 0$ so that $Q(\alpha) = P(\alpha) = 1$ and Eqn. (45) agrees precisely with the results of section 5. (Unexpectedly, agreement is exact because the integral expression of Eqn. (26) in the case without evaporation has an unusual property; namely, that steepest-descents estimation gives its accurate value.) On the other hand, $Q(\alpha) < 1$ for positive α so that decay in number density with height for droplets of initial fall speed V_0 occurs with an e-folding distance augmented from D/V_0 to

$$\frac{D}{V_0 Q(\alpha)} \qquad (47)$$

the presence of the multiplier $P(\alpha)$ making little difference to this conclusion.

It is interesting that evaporation sets an upper limit on height, which Eqn. (43) specifies as increasing with initial fall speed V_0; while, within that upper limit, gravity brings about an exponential decay in number density with an e-folding distance given by Eqn. (47) which responds to increase in V_0 by steadily decreasing - albeit rather more slowly in consequence of evaporation.

6. Concluding remarks

To draw conclusions in full detail from this model will take time, but a first attempt at using it (as proposed in section 1) to infer a tentative extrapolation of existing knowledge on ocean spray to much higher wind speeds has already been made (Lighthill). It suggests that, at doubled speed, spray mass may increase by a factor of 3 over and above the expected big rise in spray generation. Such a suggestion, in relation to the thermodynamics of tropical cyclones, may possibly mean that acceleration to extreme wind speeds produces a so greatly increased mass of spray per unit horizontal area that 'spray cooling' grows in effectiveness even as relative humidity approaches 1 at the eyewall. This could create, in turn, a 'self-limiting' effect (Lighthill 1997) in any possible influence of global warming on tropical cyclone intensitites.

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