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The Growth of Cloud droplets by Coalescence

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Abstract. An attempt is made to obtain an estimate of the percentage of small droplets swept out of the path of a large drop. The method followed is along the lines of a similar treatment by G. I. Taylor and developed by Glauert for droplets approaching a cylinder and an aerofoil. Similar computations were also made by Langmuir and Blodgett in U. S. A. using a differential analyser ; but neglecting the discrete sizes of small droplets. This has been taken into account in the present work.

Sets of trajectories are drawn for droplets approaching a sphere having the dimensions of a large cloud drop. Using these trajectories new values of the percentage catch have been computed in a number of cases.

1. *Introduction.*

Estimates of the growth of cloud droplets hav beene made in recent years by Findeisen[1] and Schumann[2] assuming that collision of drops leads to coalescence. Their work, however, suffers from the limitation that no account is taken of the deflection of one drop approaching another by the flow of air relative to the drops. Langmuir[3] took account of this factor but did not take into consideration the finite sizes of small droplets in working out the 'collection efficiency' of large drops.

Owing to the streamline pattern round a spherical drop, it is able to sweep up only a fraction of small droplets in its downward path. The amount swept up depends also on the size of small droplets that come in the way of a large drop. We have, therefore, obtained new values of the percentage catch for drops of three different sizes approaching a large drop.

2. *The 'Percentage Catch' of small droplets made by a large drop.*

The method used in determining the percentage catch was similar to that used by Taylor and Glauert in connection with water droplets approaching an aerofoil.

We consider two drops A and B (Fig. 1) falling independently with terminal velocities V_1 and V_2. The drop B when distant from A approaches it with velocity,

$$S = V_1 - V_2 \qquad (2\cdot1)$$

The 'free-stream' or undisturbed velocity of air at a large distance from A is given by,

$$U = V_1 \qquad (2\cdot2)$$

Let P represent the centre of the small droplet (B) such that its trajectory from P provides tangential contact with the large drop (A). If the droplet (B) be further displaced from the axis of symmetry it would be so deviated as not to touch the large drop. The 'Percentage Catch' (M) made by large drop A is then defined by,

$$M = \frac{OP^2}{R^2} . 100 \qquad (2\cdot3)$$

it being assumed that contact leads to coalescence. Such a definition provides for a possible catch greater than 100°/$_0$ as would be the case for $(R + a) > OP > R$, where 'a' is the radius of the smaller drop.

3. *The equations of motion of the small drop.*

The trajectory of a small drop such as B (Fig. 1) was obtained by integrating its equations of motion using the method of finite differences.

To derive the equations of motion the axes of symmetry as shown in Fig. 1 have been rotated as in Fig. 2. The equations of motion of a small drop at any point P in a steady stream (see Fig. 2) moving relative to a fixed larger drop are then given by,

$$\tfrac{4}{3}.\ \pi\sigma.\ a^3.\ \frac{du_1}{dt} = \tfrac{1}{2}\ \pi a^2.\ C_D.\ \rho.\ q.\ (u - u_1)$$

$$\tfrac{4}{3}.\ \pi\sigma.\ a^3.\ \frac{dv_1}{dt} = \tfrac{1}{2}\ \pi a^2\ C_D.\ \rho.\ q.\ (v - v_1) \qquad (3{\cdot}1)$$

where u_1, v_1 = Components of drop velocity at a point x, y of its trajectory, along axes ox, oy fixed to the large drop.

u, v = Components of the air velocity relative to the large drop at a point x, y on the streamline, along the axes ox, oy.

$q = \{(u_1 - u)^2 + (v_1 - v)^2\}^{\frac{1}{2}}$

a = Radius of small drop

C_D = Coefficient of drag on small drop

ρ = Density of air

σ = Density of drop

For small drops obeying Stoke's Law the above equations reduce to,

$$k.\frac{du_1}{dt} = u - u_1$$

$$k.\frac{dv_1}{dt} = v - v_1 \qquad (3{\cdot}2)$$

where $K = \frac{2}{9}.\ \frac{\sigma}{\eta}\ .\ a^2$

η = Coefficient of viscosity of air.

The above equations may be made non-dimensional in terms of a standard velocity and length. We take as our standard velocity the free stream velocity of air. This is equal to the terminal velocity of the large drop. The standard length is taken as the radius of the large drop.

Expressing each of the variables in 3·2 in terms of the above standards, the equations become.

$$k.\ \frac{du_1}{dt} = u - u_1$$

$$k\ \frac{dv_1}{dt} = v - v_1 \qquad (3{\cdot}3)$$

where u_1, v_1 etc., are now expressed in terms of U
x, y etc., are expressed in terms of R.

$$\text{and, } k = \frac{2}{9}\ .\ \frac{\sigma}{\eta}\ .\ \frac{a^2U}{R} \qquad (3{\cdot}4)$$

We shall consider for the present only the case in which the smaller of the two drops obeys Stoke's Law. The size of the large drop is not limited by this except as provided below.

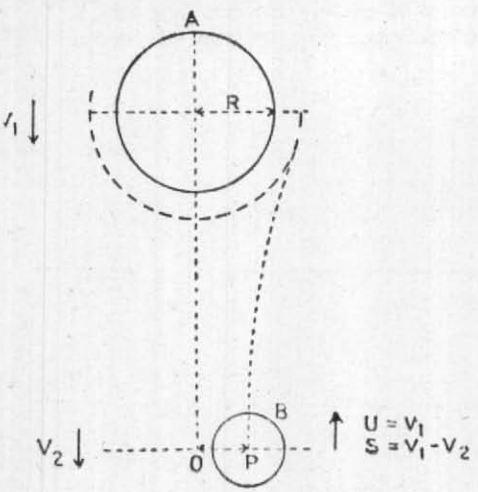


Fig. 1.

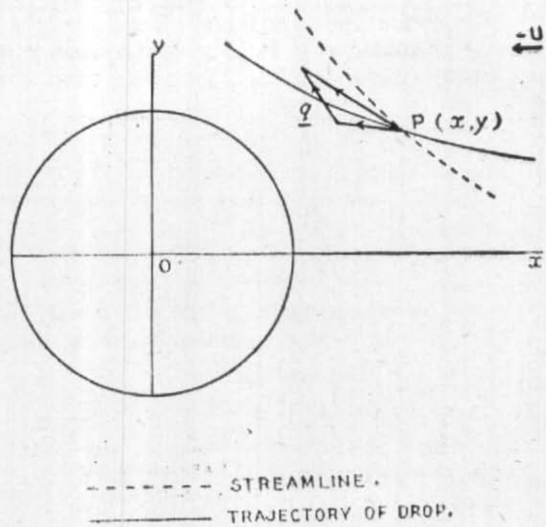


Fig 2.

4. Significance of the parameter k.

For drops obeying Stoke's Law the terminal velocity varies as the square of the radius. The 'free stream' velocity of air may thus be expressed in terms of the radius of the large drop,

$$U = 1{\cdot}313 \times 10^{6} \cdot R^{2} \qquad (4{\cdot}1)$$

The variation of 'η' with temperature is small. Its mean value between $+15°c$ and $-37°c$ is given by,

$$\eta = 1{\cdot}66 \,.\, 10^{-4} \text{gm. cm}^{-1}\text{. Sec}^{-1} \qquad (4.2)$$

and $\sigma = 1{\cdot}00$ gm. cm^{-3}

Putting the above values in 3·4 we get,

$$k = 1{\cdot}759 \,.\, 10^{9} .\ \left(\frac{a}{R}\right).\ (aR^{2}) \qquad (4.3)$$

The relation between k, 'a' and 'R' for three arbitrarily chosen values of $\frac{a}{R}$ is shown in the following table.

k	2·00		1·00		0·75		0·5		0·25		0·10	
a/R	R (μ)	a (μ)	R (μ)	a (μ)	R (μ)	a (μ)	R (μ)	a (μ)	R (μ)	a(μ)	R (μ)	a(μ)
0·1	48·5	4·9	38·5	3·9	34·9	3·5	30·5	3·1	24·2	2·4	17.8	1·8
0·2	38·5	7·7	30·5	6·1	27·7	5·6	24·2	4·9	19·2	3·9	14·2	2·8
0·3	23·3	7·0	18·5	5·5	16·8	5.0	14·7	4·4	11·6	3·5	8·6	2·6

The results are shown graphically in Fig. 3.

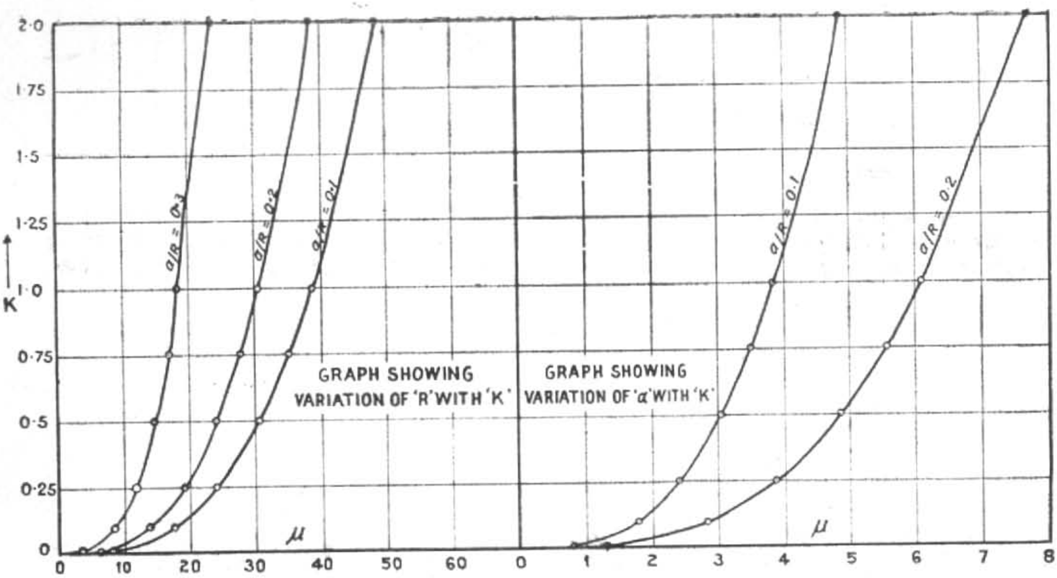


FIG. 3.

It is apparent that apart from being a function of drop sizes as shown above, the parameter 'k' is also a measure of the inertia of the smaller drop. For smaller values of 'k' the trajectories show large deviation, while for larger values of k, they remain almost undeviated. The functional relationship between 'k' and the 'percentage catch' is established by determining the latter for different values of 'k'.

5. *Integration of the equations of motion.*

The equations of motion in 3·3 were integrated using finite differences and the values of x_1 y_1 thus obtained were used to trace the trajectory of a drop.

For a perfect fluid the values of u, v at any point a head of a sphere are given by,

$$u = -\left\{ 1 + \frac{1}{2r^3} \left(1 - \frac{3x^2}{r^2} \right) \right\}$$

$$v = \frac{3}{2} \cdot \frac{xy}{r^5}$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ (5·1)

(Aerodynamic Theory—Durand, Vol. 1)

The above are expressed in non-dimensional units and refer to a diametral plane of the sphere. Strictly, trajectory computations should be started from infinity where the streamlines become parallel and the small drop is undisturbed, as Langmuir has done. However, the variation of u, v is very small for $x > 3$ which has been chosen as the starting point for the computations.

We have also to define the condition of the small drop at $x=3$, and two assumptions have been made according to the size of the small drop *i.e.* according to the value of k. When the drop is very small indeed, its motion relative to the air in its neighbourhood will differ only infinitesimally from its terminal velocity V_2, that is, the drop has fully responded to the very small acceleration of the air at $x > 3$. Formally.

$$u_1 = u - V_2$$
$$v_1 = v \qquad \text{for } k \leqslant 0{\cdot}5 \text{ at } x=3$$

In non-dimensional units the above becomes,

$$u_1 = u - \left(\frac{a}{R}\right)^2$$
$$v_1 = v$$

For larger drops $(k \geqslant 0{\cdot}5)$ we assume that the motion of the drop relative to the air is the difference in the terminal velocities of the two drops, that is the very small acceleration of the air for large x has left the droplet unaffected. We then have,

$$u_1 = V_1 - V_2$$
$$\qquad \text{for } k \geqslant 0{\cdot}5 \text{ at } x=3$$
$$v_1 = 0$$

In non-dimensional units,

$$u_1 = -1 + \left(\frac{a}{R}\right)^2$$
$$v_1 = 0$$

The above two boundary conditions represent two extremes and in reality the motion of the drop lies somewhere between these above values. The inclusion of $k=0{\cdot}5$ in both categories gives an opportunity of assessing the effect of the change in boundary conditions.

The change in velocity and position of a drop during a time interval Δt, is given by the following equations,

$$U_{1x_{p+1}} = U_{1x_p} - \frac{1}{k} \cdot \left(U_{x_p} - U_{1x_p} \right). \ \Delta t.$$

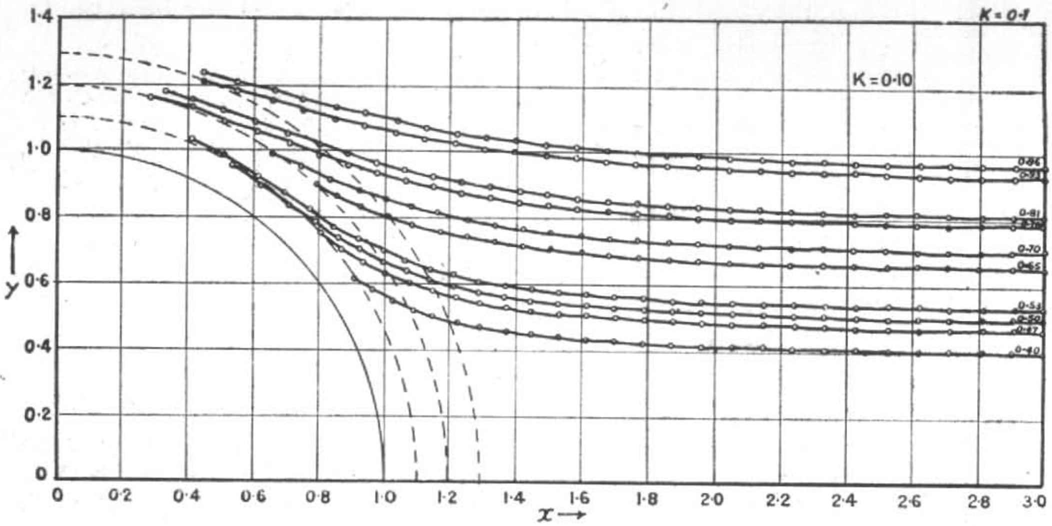
$$V_{1x_{p+1}} = V_{1x_p} - \frac{1}{k} \cdot \left(V_{x_p} - V_{1x_p} \right). \ \Delta t. \qquad (5{\cdot}2)$$

and,

$$x_{p+1} = x_p + u_1 \Delta t.$$
$$y_{p+1} = y_p + v_1 . \ \Delta t. \qquad (5{\cdot}3)$$

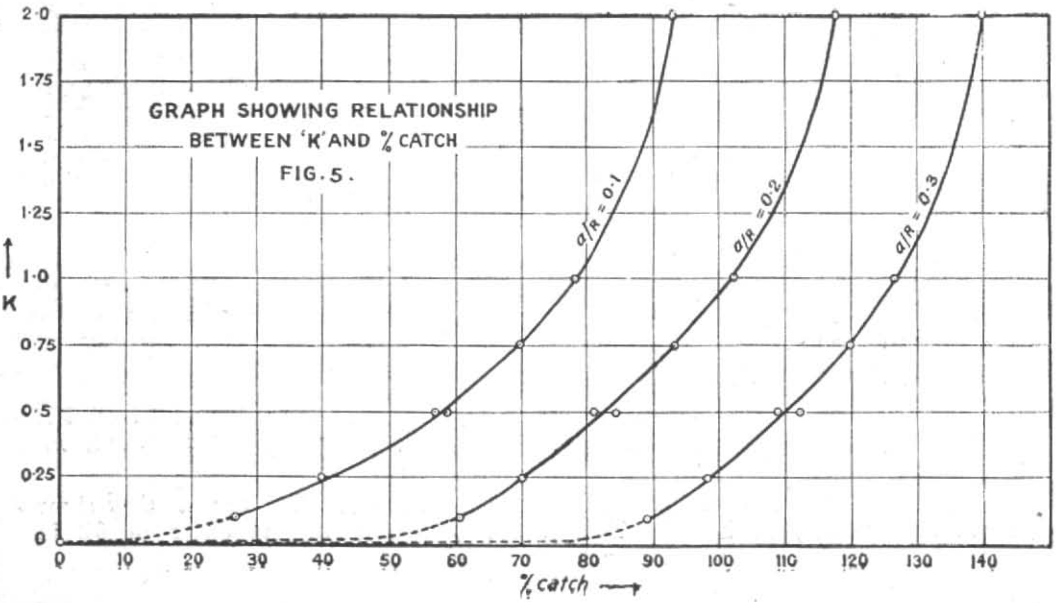
Trajectories were obtained for several values of 'k'. A typical set for $k=0{\cdot}1$ is shown in fig. 4. They have been drawn for three values of $a/R=0{\cdot}1, 0{\cdot}2, 0{\cdot}3$ respectively, as shown by dotted circles round the large drop.

The shape of the k—% catch curves is shown in Fig. 5. As expected they show a small discontinuity at $k=0{\cdot}5$. This however, is only of the order of 5% in ' percentage catch ' so that the mean value could be taken.



TRAJECTORIES OF DROPLETS APPROACHING A LARGE DROP.

FIG. 4.



GRAPH SHOWING RELATIONSHIP BETWEEN 'K' AND % CATCH

FIG. 5.

6. *Limitations of the above method.*

(*a*) The time intervals had to be made small to obtain accurate trajectories. The smallest time interval used was of the order of 0·1. Thus, $\triangle t = 0{\cdot}1$ was used for all trajectories with $k \leqslant 0{\cdot}5$, and was opened upto $\triangle t = 0{\cdot}3$ for larger 'k' in the early intervals, tests having shown this to be satisfactory.

(*b*) The above treatment is only valid for small drops in the Stoke's Law regime. It has also been assumed that the air flow ahead of a large drop is that of a perfect fluid. The streamlines for viscous flow ahead of a sphere are more deviated than in a perfect fluid, consequently the catch will be less than that for a perfect fluid. The air flow ahead of a large drop is in reality between the limiting cases of perfect and viscous fluid flow. Within the boundary layer the flow is likely to correspond to viscous flow while beyond the boundary layer the flow would be more like that of a perfect fluid. Further work is now in progress to take into account the viscosity of air.

7. *Discussion of Results.*

Langmuir computed the percentage catch for different values of 'k' both for 'perfect fluid' flow and viscous flow. He does not appear, however, to have computed the catch for different values of a/R, whereas, as we see from Fig. 5, the catch is markedly dependent on a/R as well as on 'k'. We have chosen three values of a/R and computed trajectories for each. It is realised, however, that a/R cannot be indefinitely increased, for if the small droplets be of size comparable to that of the large drop the air flow would no longer be that ahead of a single sphere. It has not been possible to determine the maximum possible value of a/R such that the air flow remains relatively undisturbed but it is thought reasonable to go as far as a/R = 0.3.

The following tables show the difference between values of percentage catch obtained by Langmuir and those obtained by us :—

k	% catch Langmuir	%catch computed a/R = 0·1	0·2	0·3
0·0833	0	24·2	58·5	88·5
0·100	1·0	26·5	59 5	89·00
0·25	11·00	39·8	69·7	100·00
0·50	24·90	57·5	82·5	111·50
0·75	35·50	69·75	92·5	119 90
1·00	44·50	78·5	101·0	126·50
2·00	64·00	93·0	117·25	140·50

It was shown by Taylor that for droplets approaching a cylinder there was a critical value of 'k' beyond which no droplets could touch the cylinder. He assumed that streamlines just ahead of the stagnation point were rectangular hyperbolas such that the flow could be represented by,

$$u = -cx$$
$$v = cy \qquad (7{\cdot}1)$$

Putting the above in the equations of motion, the condition for no drops touching the cylinder was,

$$k_0 . C \leqslant \tfrac{1}{4}$$

Where $k_o = \frac{2}{9} \cdot \frac{\sigma}{\eta} \cdot a^2$ (7.2)

for flow past a cylinder,

$$C = \frac{2U}{R}$$

Where U = Free stream velocity of air
R = Radius of cylinder

Therefore, the critical condition is,

$$2\left(\frac{2}{9} \cdot \frac{\sigma}{\eta} \cdot \frac{a^2}{R} \cdot U\right) \leqslant \tfrac{1}{4}$$

$$\text{or } k \leqslant \tfrac{1}{8} \qquad (7{\cdot}3)$$

For flow past a sphere Langmuir obtained the critical value of k to be 0·0893.

In our method the k — % catch curves so far obtained by drawing trajectories upto k=0·1 suggest very small values of critical k. Indeed, the curves all tend to pass through the origin suggesting that the percentage catch becomes nil as 'k' approaches zero.

8. *Acknowledgement.*

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APPENDIX.

Critical trajectory data.

	y_o/R for different a/R.			% Catch=100 $(y_o/R)^2$		
k	0·1	0·2	0·3	0·1	0·2	0·3
2·00	0·965	1·085	1·185	93·12	117·72	140·4
1·00	0·885	1·010	1·125	78·32	102·02	126·56
0·75	0·835	0·965	1·095	69·72	93·16	119·90
0·50 (A)	0·765	0·920	1·060	58·52	83·84	112·36
0·50 (B)	0·755	0·900	1·045	57·00	81·00	109·2
0·25	0·630	0·835	1·000	39·69	69·74	100·0
0·10	0·515	0·780	0·945	26·52	60·84	89·30

The critical trajectory in each case was determined by interpolating between a trajectory that just makes contact and another that just misses the large drop. The trajectories were so spaced that errors due to this could not exceed 5% in the percentage catch.

REFERENCES:

1. Findeisen, *Met. Zeit*, 56, (1939).
2. Schumann, *Q. J. R. Met. Soc.*, 285, (1940).
3. Langmuir, I., *Jour. Met.*, 5, 5, (1948).
4. Taylor, G. I., *Aero Research Committee, Rep. & Memoranda*, 2024.
5. Glauert, M., *Aero Research Committee, Rep. & Memoranda*, 2025.