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Vertical Velocity and Rainfall by Convergence due to Latitude effect

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ABSTRACT. Convergence in northward motion without north-south gradient of velocity or due to variation of Coriolis parameter in geostrophic equation is termed as latitudinal convergence. In this paper the orders of vertical velocity and rainfall arising from such convergence are computed and it is known that they are small in tropics.

1. Convergence due to northward motion

The equation of continuity in polar co-ordinates is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial \lambda}\left(\rho \dot{\lambda}\right)+\frac{\partial}{\partial \phi}\left(\rho \dot{\phi}\right)+\frac{\partial}{\partial r}\left(\rho \dot{r}\right)$$

$$-\rho \dot{\phi}\tan\phi+\frac{2}{r}\rho \dot{r}=0 \qquad \ldots\ldots(1)$$

where ϕ is the latitude, λ the longitude, and r the distance from the centre of the co-ordinate system (*i.e.*, centre of the earth) and ρ the density of air. It is the general practice to measure air motion at any point in a system of rectangular co-ordinates fixed in the earth at that point, with the xy plane tangential to the earth's surface and z co-ordinate along the vertical.

The relationships between the two systems of co-ordinate are given by

$$dx=r.\cos\phi.d\lambda$$
$$dy=rd\phi$$
$$dz=dr \qquad \ldots\ldots\ldots(2)$$

Also

$$u=\frac{dx}{dt}=r\cos\phi.\ \dot{\lambda}$$
$$v=\frac{dy}{dt}=r\dot{\phi}$$
$$w=\frac{dz}{dt}=\dot{r} \qquad \ldots\ldots\ldots\ldots(3)$$

where u, v, w are the velocities towards East, North and in the vertical at any point.

Substituting in equation (1) we get[1]

$$\frac{\partial \rho}{\partial t}+\frac{1}{r\cos\phi}\ \frac{\partial}{\partial \lambda}\left(\rho u\right)+\frac{1}{r}\frac{\partial}{\partial \phi}\left(\rho v\right)$$

$$+\frac{\partial}{\partial r}\left(\rho w\right)-\frac{\rho v}{r}\tan\phi+\frac{2}{r}\rho w=0 \qquad ..(4)$$

This equation shows that apart from gradients of u along latitude circle, of v along meridional circle, vertical motion arises also from the very motion along meridional circle (northward motion of air) as seen from the terms $\frac{\rho v}{r}\tan\phi$ and $\frac{2}{r}\rho w$. Thus even if all the other terms were zero (as for example with a constant v and u at all points), $\frac{2}{r}\rho w=\frac{\rho v}{r}\tan\phi-\frac{\partial \rho}{\partial t}$ indicating that vertical motion is caused by the very nature of the northward motion (unless compensated by density variations with time), even if there be no wind gradient along latitude or longitude. This, however, arises from the nature of co-ordinate system in which air motion is being specified and it can be shown by specifying the air motion at all points in a single system of rectangular co-ordinates that the terms $\frac{\rho v}{r}\tan\phi$ and $\frac{2}{r}\rho w$ arise out of gradients of velocity.

2. Vertical velocity

From equaton (4) it is seen that the northward moving air has the same effect as convergence and southward moving air as divergence. This may be deemed as one type of convergence due to latitude effect. The vertical velocities arising therefrom are computed below. Since our object is to investigate the effect of $\frac{\rho v}{r}\tan\phi$ on w, we shall exclude other effects by putting

$$\frac{\partial \rho}{\partial t}=\frac{\partial}{\partial \lambda}\left(\rho u\right)=\frac{\partial}{\partial \phi}\left(\rho v\right)=0$$

Hence

$$\frac{\partial}{\partial r}\Big(\rho w \Big) + \frac{2}{r}\rho w - \frac{\rho v}{r}\tan\phi = 0 \quad \ldots (5)$$

Multiplying by r^2

$$r^2\frac{\partial}{\partial r}(\rho w) + 2r\rho w - r\rho v \tan\phi = 0$$

$$\therefore \frac{\partial}{\partial r}\Big(r^2\rho w \Big) = r\rho v \tan\phi \quad \ldots\ldots\ldots (6)$$

$$\therefore \int_a^{a+z} \frac{\partial}{\partial r}\Big(r^2\rho w\Big)dr = \int_a^{a+z} r\rho v \tan\phi \, dr \quad \ldots (7)$$

a is the radius of the earth and z is the vertical height above the surface of the earth. For simplicity we take that v is constant at all heights. Hence,

$$\Big[r^2\rho w \Big]_a^{a+z} = v\tan\phi \int_a^{a+z} r\rho dr$$

Now, $dp = -g\rho dr$, where p is the pressure.

$$\therefore \Big[r^2 \rho w \Big]_a^{a+z} = -\frac{v\tan\phi}{g}\int_a^{a+z} r dp$$

At the surface of the earth $(r=a)$ vertical velocity is zero.

Hence,

$$(a+z)^2 \rho_z w_z = -\frac{v\tan\phi}{g}\int_a^{a+z} r dp.$$

It will be seen later that we are here concerned with heights up to four miles above the earth's surface *i.e.*, $z < 4$ miles while a is nearly equal to 4000 miles. r varies between about 4000 and 4004 and may be treated as a constant equal to 'a' to a high degree of approximation. We may also write a^2 is nearly equal to $(a+z)^2$. The error on account of this is 0.2%. So,

$$a^2 \rho_z w_z = -\frac{v\tan\phi}{g}\, a \int_a^{a+z} dp \quad \ldots (8)$$

or

$$w_z = \frac{v\tan\phi}{g a \rho_z}\Big(p_0 - p_z\Big) \quad \ldots\ldots (9)$$

where p_0 is the pressure at the surface of the earth and p_z is the pressure at height of z above.

The following table gives the vertical velocities at various pressure levels in different latitudes when $v=30$ knots. Saturated air column with saturated adiabatic lapse rate and $p=1000$ mb and temperature of 27°C at the base of the column is considered. $g=980$ *cm* Sec^{-2}, $a=$ 3438 nautical miles. A constant value of $v=$ 30 knots at all levels up to 500 mb has been chosen to illustrate the orders of magnitude of the resulting vertical velocities and for simplifying computation. For any other type of velocity distribution the resulting vertical velocities can also be computed. But as northward velocities usually met with are less than 30 knots , the vertical velocities given here may be considered as probable upper limits.

TABLE 1.

w_z *in cm/sec*

P_z (*mb*)	Latitude— ϕ 5°	10°	20°	30°	35°
900	0.02	0.04	0.09	0.14	0.17
800	0.05	0.09	0.19	0.30	0.37
700	0.08	0.15	0.32	0.51	0.62
600	0.12	0.24	0.49	0.78	0.94
500	0.17	0.35	0.72	1.14	1.38

On account of $\tan\phi$ in equation (9) the vertical velocity increases with latitude. But it will be seen that upto 35° the vertical velocities are too small to be of consequence. We have considered saturated atmosphere here but the vertical velocities will not be different in order of magnitude if the air is unsaturated. If in such a case the condensation level is 600 metres, with a vertical velocity of 0.17 cm/sec the air takes about 100 hours to rise to the condensation level from the ground.

3. Rate of rainfall

We may roughly compute the maximum rate of precipitation that can result from latitudinal convergence alone in the case under consideration. Let us consider a volume bounded by the surfaces r, $r+dr$, ϕ, $\phi+d\phi$, λ and $\lambda+d\lambda$ (see Fig. 1).

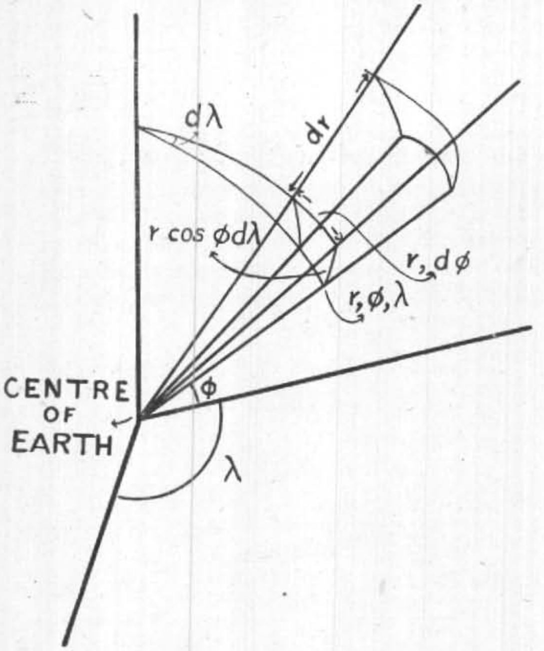

Fig. 1

The rate of increase of moisture content in this volume is the difference between the moisture entering and leaving through these surfaces. It can be shown that the rate of increase is

$$- \left\{ \frac{1}{r\cos\phi} \frac{\partial}{\partial\lambda} \left(\rho q u\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\rho q v\right) + \frac{\partial}{\partial r}\left(\rho q w\right) - \frac{\rho q v}{r}\tan\phi + \frac{2}{r}\rho q w \right\} r^2 \cos\phi .\, d\phi .\, d\lambda \;\; dr$$

where q is the specific humidity of the air.

Assuming steady state *i.e.*, $\frac{\partial\rho}{\partial t} = 0$, and substituting from the resulting equation of continuity, the above expression becomes

$$- \left\{ \frac{\rho u}{r\cos\phi} \frac{\partial q}{\partial\lambda} + \frac{\rho v}{r} \frac{\partial q}{\partial\phi} + \rho w \frac{\partial q}{\partial r} \right\} r^2 \cos\phi .\, d\phi .\, d\lambda .\, dr$$

We may assume that q does not vary with λ or ϕ which is equivalent to assuming that the airmass is homogeneous in the horizontal. This expression hence reduces to

$$- \rho w \frac{\partial q}{\partial r} .\, r^2 \cos\phi .\, d\phi .\, d\lambda .\, dr$$

Since the air is saturated, this moisture condenses and we assume that it is immediately precipitated over an area of $a^2 \cos\phi .\, d\phi .\, d\lambda$ at the ground (From Fig. 1 it can be seen that the area at the ground intercepted by the surfaces r, $r + dr$, ϕ, $\phi + d\phi$, λ and $\lambda + d\lambda$ is $a^2 \cos\phi .\, d\phi . d\lambda$). Hence the total rate of rainfall per unit area at the ground is given by

$$- \int \frac{r^2}{a^2} \rho w \frac{\partial q}{\partial r} dr,$$

the expression being integrated from the ground to a height z, where either the vertical velocity is zero or the amount of moisture content is very small and hence the contribution to rainfall is likely to be negligible.

$$- \int \frac{r^2}{a^2} \rho w \frac{\partial q}{\partial r} dr = - \int_0^z \frac{(a+z)^2}{a^2} \rho w \frac{\partial q}{\partial z} dz$$

is nearly equal to $- \int_0^z \rho w \frac{\partial q}{\partial z} dz.$

Upto the height of 4 miles above the ground that we are concerned $\frac{(a+z)^2}{a^2}$ is nearly equal to 1 and the error on this account as shown earlier is only 0.2 %.

$$\text{Rate of rainfall} = - \int_0^z \rho w \frac{\partial q}{\partial z} dz$$

$$= - \int_0^z \rho w \frac{\partial q}{\partial p} dp,$$

it being understood that the height z corresponds to some pressure level. By equation (9)

$$\rho_z w_z = \frac{v \tan\phi}{ga} \left(p_0 - p_z \right)$$

$$\therefore - \int_0^z \rho w \frac{\partial q}{\partial z} dz = - \int_0^z \frac{v \tan\phi}{ga} \left(p_0 - p\right) \frac{\partial q}{\partial p} dp$$

$$= - \frac{v \tan\phi}{ga} \int_0^z \left(p_0 - p\right) \frac{\partial q}{\partial p} dp \quad \ldots\ldots\ldots(10)$$

Now $\int_{o}^{z} \left(p_o - p\right) \frac{\partial q}{\partial p} . dp$

$$= p_o \Big[q \Big]_o^z - \int_o^z p \frac{\partial q}{\partial p} dp$$

$$= p_o \Big[q \Big]_o^z - \Big[pq \Big]_o^z + \int_o^z q dp$$

$$= p_o \left(q_z - q_o\right) - \left(p_z q_z - p_o q_o\right) + \int_o^z q dp$$

$$= \left(p_o - p_z\right) q_z + \int_o^z q dp$$

Rate of rainfall

$$= -\frac{v \tan \phi}{ga}\left[(p_o - p_z)q_z + \int_o^z q dp\right] \quad .. \quad (11)$$

We shall compute the rate of rainfall from a saturated column between 1000 and 500 mb (with saturated adiabatic lapse rate) at the latitude of 35° with a northward velocity of 30 knots.

$$(p_o - p_z)q_z = 4750 \ mb. \ gm./kg.$$

$$\int_o^z q dp = -8250 \ mb. \ gm./kg.$$

Hence the rate of rainfall will be 0.02 cm/hr, which is very small.

The effect of convergence due to northward motion on lapse rate is computed below. If γ is the lapse rate, then in unsaturated air,

$$\frac{d\gamma}{dt} = \frac{\partial w}{\partial z}(\gamma_d - \gamma) \ldots\ldots\ldots\ldots\ldots (12)$$

where γ_d is the dry adiabatic lapse rate. Let $\gamma = 6°$C/km. The value of w at 500 mb level calculated previously for saturated air will not be much different even for unsaturated air. The height of the 500 mb surface under the conditions stated earlier is 5896 metres and the value of w at that level at 35°N is 1.38 cm/sec. Substituting in equation (12)

$$\frac{d\gamma}{dt} = 0.034 \ °\text{C/km. hr.}$$

In a period of ten hours the lapse rate will change from 6° C/km to 6.34° C/km which is hardly a material change.

4. Convergence in geostrophic motion

It can be shown that when the wind is geostrophic

$$\frac{1}{r \cos \phi} \frac{\partial}{\partial \lambda}(\rho u) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho v) = -\frac{2\rho v_g \cot 2\phi}{r}$$

where v_g is the north south component of the geostrophic wind. Substituting in equation (4) and putting $\frac{\partial \rho}{\partial t} = 0$,

$$\frac{\partial}{\partial r}(\rho w) + \frac{2}{r}\left(\rho w\right) - \frac{2 \rho v_g \cot 2\phi}{r} - \frac{\rho v_g}{r} \tan \phi = 0 \quad \ldots\ldots (13)$$

This equation shows that when the wind is geostrophic, apart from the effect of the term $\frac{\rho v_g}{r} \tan \varphi$, convergence results in northward motion and divergence in southward motion due to the term $\frac{2 \rho v_g \cot 2\phi}{r}$. This is due to variation in the Coriolis parameter with latitude on account of which the same pressure gradient causes a smaller geostrophic wind in higher latitudes. The effect of

$$\frac{2\rho v_g \cot 2\phi}{r}$$

in equation (13) is much larger than $\frac{\rho v_g}{r} \tan \phi$ in low latitudes. Integrating equation (13)we get

$$w_z = \left[\frac{v_g \tan \phi}{ga \ \rho_z}(p_o - p_z) + \frac{2 v_g \cot 2\phi}{ga \ \rho_z}(p_o - p_z)\right] \quad \ldots\ldots (14)$$

Since we have already computed the vertical velocity due to tan ϕ term which has been found to be small, we give below the vertical velocities due to the $\cot 2\phi$ term. v_g is taken as 30 knots.

$$w'_z = \frac{2 v_g \cot 2\phi}{ga \ \rho_z}(p_o - p_z)$$

TABLE 2

w_z' *in cm/sec*

$P_z(mb)$	Latitude—ϕ 5°	10°	20°	30°	35°
900	2.7	1.3	0.6	0.3	0.2
800	6.1	2.9	1.3	0.6	0.4
700	10.0	4.9	2.1	1.0	0·6
600	15.3	7.4	3.2	1.6	1.0
500	22.3	10.8	4.7	2.3	1.4

The rate of rain falling from a saturated column between 1000 mb and 500 mb due to $\frac{2\, v_g \cot 2\phi}{ga\, \rho_z}\left(p_o - p_z\right)$ is given below.

TABLE 3

Latitude	5°	10°	20°	30°	35°
Amount of rain (cm/hr)	0.35	0.17	0.07	0.04	0.02

These tables show that the contribution to vertical velocity or rainfall from the cot 2ϕ term in latitudes higher than 15° is very small.

The small magnitude of the convergence due to latitude effect has been recognised by previous workers[2]. This note gives an estimate of the vertical velocities and probable rate of precipitation arising therefrom. It may here be pointed out that many workers[3] have given the divergence in geostrophic wind as— $\frac{\rho v_g \cot\phi}{r}$ while we get the divergence due to geostrophic effect only as $-\frac{2\rho v_g \cot 2\phi}{r}$, though the total divergence in geostrophic wind as given by— $\frac{2\rho v_g \cot 2\phi}{r} - \frac{\rho v_g \tan\phi}{r}$ is equal to $-\frac{\rho v_g \cot\phi}{r}$.

5. Noncommutability of differentiation

The derivation of divergence in geostrophic wind is generally given as follows:
The equations of geostrophic wind are

$$2\omega\, \rho v \sin\phi = \frac{\partial p}{\partial x} \qquad \ldots\ldots\ldots(15)$$

$$\text{and } 2\omega\, \rho u \sin\phi = -\frac{\partial p}{\partial y} \qquad \ldots\ldots(16)$$

The co-ordinate system has its origin fixed in the earth at the instantaneous position of the air parcel and the x and y axes point respectively to east and north at that point and z axis points vertically upwards.

In terms of polar co-ordinates with the centre as origin, the differential relations (see equation 2) connecting the above x, y, z axes with the r, λ, ϕ system are

$$\left.\begin{aligned} dx &= r\cos\phi.\, d\lambda \\ dy &= r.\, d\phi \\ dz &= dr \end{aligned}\right\} \qquad \ldots.(17)$$

Using these relations, we get from (15) and (16)

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\rho v}{r}\cot\phi$$

$$= \frac{1}{2\omega \sin\phi}\left[\frac{\partial^2 p}{\partial y.\,\partial x} - \frac{\partial^2 p}{\partial x.\partial y}\right]$$

It is generally taken that $\frac{\partial^2 p}{\partial y.\,\partial x} = \frac{\partial^2 p}{\partial x.\partial y}$

and hence $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = -\frac{\rho v \cot\phi}{r}$

However, it should be noted that x, y, z co-ordinate system defined above changes from point to point and is not parallel between any two points.

By using relations (17) it can be shown that if F is any function,

$$\frac{\partial F}{\partial x} = \frac{1}{r}\cos\phi\frac{\partial F}{\partial \lambda}$$

$$\frac{\partial F}{\partial y} = \frac{1}{r}\frac{\partial F}{\partial \phi}$$

$$\text{and } \frac{\partial F}{\partial z} = \frac{\partial F}{\partial r}$$

and also that

$\frac{\partial^2 F}{\partial x.\partial y}$ is not $= \frac{\partial^2 F}{\partial y.\partial x}$, $\frac{\partial^2 F}{\partial z.\partial x}$ is not $= \frac{\partial^2 F}{\partial x.\partial z}$

and $\frac{\partial^2 F}{\partial z.\partial y}$ is not $= \frac{\partial^2 F}{\partial y.\partial z}$

but

$$\left.\begin{aligned} \frac{\partial^2 F}{\partial y.\partial x} &= \frac{\partial^2 F}{\partial x.\partial y} + \frac{\tan\phi}{r}\frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial x.\partial z} &= \frac{\partial^2 F}{\partial z.\partial x} + \frac{1}{r}\frac{\partial F}{\partial x} \\ \text{and } \frac{\partial^2 F}{\partial y.\partial z} &= \frac{\partial^2 F}{\partial z.\partial y} + \frac{1}{r}\frac{\partial F}{\partial y} \end{aligned}\right\} \quad ..(18)$$

This difference does not seem to have been recognised in meteorological literature. It may be mentioned that equalities like

$$\frac{\partial^2 F}{\partial x.\partial y} = \frac{\partial^2 F}{\partial y.\partial x}, \quad \frac{\partial^2 F}{\partial z.\partial x} = \frac{\partial^2 F}{\partial x.\partial z}$$

etc., are frequently used *e.g.*, in derivation of thermal wind, where strictly speaking equation (18) has to be used. However, the magnitude of the above error is in general likely to be small because of the occurrence of the radius of the earth in the denominator of the last term in equation (18). By taking account that

$$\frac{\partial^2 F}{\partial y.\partial x} - \frac{\partial^2 F}{\partial x.\partial y} = \frac{\tan\phi}{r}\frac{\partial F}{\partial x}$$

it can be shown that

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = -\frac{2v_g\rho}{r}\cot 2\phi$$

for geostrophic winds.

The very small magnitude of convergence due to latitude effect $\frac{v}{r}\tan\phi$ or $\frac{2v\cot 2\phi}{r}$ in comparison with the convergence due to wind gradient may also be judged by the following values.

TABLE 4

For v = 30 knots

Latitude	10°	15°	20°	30°	35°	
$\frac{v}{r}\tan\phi$	0.0015	0.0023	0.0032	0.0050	0.0063	hr^{-1}
$\frac{2v}{r}\cot 2\phi$	0.048	0.030	0.021	0.010	0.006	hr^{-1}

Houghton and Austin[4] prepared convergence charts from observed upper winds and found that values of 0.144 hr^{-1} were observed on many charts. In some preliminary work on computation of convergence from reported winds in India we frequently get convergence values between 0.1 and 0.2 hr^{-1}. The velocity of northward motion considered here is 30 knots which is much in excess of what is commonly observed on weather charts. We may conclude that $\frac{v}{r}\tan\phi$ is of no consequence to weather from equator up to 35° latitude. $\frac{2v}{r}\cot 2\phi$ is more effective in lower latitudes but even this may be neglected in latitudes higher than 15°. Besides, the derivation of its magnitude depends on the assumption of geostrophic wind and implies that there is no tangential acceleration, which may be inconsistent with a northward gradient of velocity.

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