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Energetic consistency of truncated models

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ABSTRACT. Energetic consistency of truncated NWP models is examined. It is shown that in the so-called 'energetically consistent' models, there are fictitious cancellations of vertical divergences of energy fluxes at each level and hence the vertical coupling of energy in these models is rather defective.

1. After the classical work of Lorenz (1960), it is well-established in the field of meteorology that certain terms or groups of terms in the vorticity and divergence equations should either be retained or dropped together to achieve "energetic consistency". This has led to an hierarchy of models beginning with complete vorticity and divergence equation model and ending with quasi-geostrophic model. By integration over closed horizontal surface, we show that the energetic consistency of the various truncated models is of more restricted type than would appear from the form which it takes for the atmospheric mass as a whole. In general, these truncated models create fictitious cancellations of vertical divergences of energy fluxes at each level and hence vertical coupling of energy in these truncated models is somewhat defective.

2. Equation of quasi-static frictionless horizontal motion is—

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k} \times \mathbf{V} + \nabla\phi = 0 \qquad (1)$$

By performing $\nabla.$ and $\mathbf{k}\,.\nabla\chi$ operations on this equation, we get the well-known divergence and vorticity equations —

$$\frac{\partial D_3}{\partial t} + \mathbf{V}_2.\nabla D_3 + \mathbf{V}_3.\nabla D_3 + \omega_3\,\frac{\partial D_3}{\partial p} + D_3{}^2 + $$
$$+ 2J\,(v_2,u_2) + 2J\,(v_3,u_3) + 2J\,(v_2,u_3) +$$
$$+ 2J(v_3,u_2) - f\zeta_2 + u_2\beta + u_3\beta + \nabla\omega_3.\frac{\partial \mathbf{V}_2}{\partial p} +$$
$$+ \nabla\omega_3.\frac{\partial \mathbf{V}_3}{\partial p} + \nabla^2\phi_1 = 0 \qquad (2)$$

$$\frac{\partial\zeta_2}{\partial t} + \mathbf{V}_2.\,\nabla\zeta_2 + \mathbf{V}_3.\nabla\zeta_2 + \omega_3\,\frac{\partial\zeta_2}{\partial p} + v_2\beta +$$
$$+ v_3\beta + fD_3 + \zeta_2 D_3 + J\left(\omega_3, \frac{\partial\chi_3}{\partial p}\right) +$$
$$+ \nabla\omega_3.\nabla\,\frac{\partial\psi_2}{\partial p} = 0 \qquad (3)$$

Following Lorenz (1960), we have used subscript 1 to indicate a physical property and subscripts 2 and 3 to indicate quantities derived from stream-function ψ_2 and velocity potential χ_3 respectively.

We multiply each term of the vorticity equation by $-\psi_2$ and each term of divergence equation by $-\chi_3$ and integrate over the earth's closed horizontal spherical surface. We do not perform vertical integration and in this respect deviate from the analysis of Lorenz (1960). We freely use the property that divergence of a horizontal vector and horizontal Jacobian of two scalers vanish when integrated over closed horizontal surface. The terms which result after some simple manipulations are shown in Table 1. We use the notation—

$$k_2 \equiv \frac{\mathbf{V}_2.\mathbf{V}_2}{2}\ ;\ k_3 \equiv \frac{\mathbf{V}_3.\mathbf{V}_3}{2}\ ;\ \mathbf{V} = \mathbf{V}_2 + \mathbf{V}_3$$

3. We now have from vorticity and divergence equations respectively,

$$\oint_s \left\{ \frac{\partial k_2}{\partial t} + \eta\nabla\psi_2.\nabla\chi_3 + \right.$$
$$\left. + \omega_3\left(\frac{\partial k_2}{\partial p} + \mathbf{V}_2.\,\frac{\partial \mathbf{V}_3}{\partial p}\right)\right\} ds = 0$$

$$\oint_s \left\{\frac{\partial k_3}{\partial t} - \eta\nabla\psi_2.\,\nabla\chi_3 + \omega_3\,\left(\frac{\partial k_3}{\partial p} + \mathbf{V}_3\cdot\frac{\partial \mathbf{V}_2}{\partial p}\right) + \right.$$
$$\left. + \mathbf{V}\cdot\nabla\,(\mathbf{V}_2.\mathbf{V}_3 + k_2 + k_3 + \phi_1)\right\} ds = 0$$

TABLE 1

$-\oint_s \chi_3$ (DIVERGENCE EQ.) ds DIVERGENCE EQUATION		$-\oint_s \psi_2$ (VORTICITY EQ.) ds VORTICITY EQUATION		SERIAL NUMBER OF GROUP	TOTAL CONTRIBUTION OF GROUP AFTER INTEGRATION OVER	
TERM OF DIVERGENCE EQ	CONTRIBUTION TO $\frac{\partial k}{\partial t}$	TERM OF VORTICITY EQ.	CONTRIBUTION TO $\frac{\partial k}{\partial t}$		CLOSED HORIZONTAL SURFACE	ENTIRE MASS OF ATMOSPHERE
$\frac{\partial D_3}{\partial t}$	$\frac{\partial k_3}{\partial t}$	$\frac{\partial \xi_2}{\partial t}$	$\frac{\partial k_2}{\partial t}$	1	$\oint_s \frac{\partial}{\partial t}(k_2+k_3)\,ds$	$\frac{\partial}{\partial t}(k_2+k_3)$
		$\mathbf{V}_2\cdot\nabla\xi_2$ $v_2\beta$	0 0	2	0	0
$u_3\beta$	0			3	0	0
$\mathbf{V}_3\cdot\nabla D_3$ D_3^2 $2J(v_3, u_3)$ $\omega_3\frac{\partial D_3}{\partial p}$ $\nabla\omega_3\cdot\frac{\partial \mathbf{V}_3}{\partial p}$	$\chi_3 D_3^2 - 2\mathbf{V}\cdot\nabla k_3$ $-\chi_3 D_3^2$ $3\mathbf{V}\cdot\nabla k_3$ $-\chi_3\omega_3\frac{\partial D_3}{\partial p}$ $\chi_3\omega_3\frac{\partial D_3}{\partial p}+\omega_3\frac{\partial k_3}{\partial p}$			4	$\oint_s (\mathbf{V}\cdot\nabla k_3+\omega_3\frac{\partial k_3}{\partial p})\,ds$ OR $\oint_s \frac{\partial}{\partial p}(\omega_3 k_3)\,ds$	0
$\mathbf{V}_2\cdot\nabla D_3$ $2J(v_2, u_3)+2J(v_3, u_2)$ $\nabla\omega_3\cdot\frac{\partial \mathbf{V}_2}{\partial p}$	$-\mathbf{V}\cdot\nabla(\mathbf{V}_2\cdot\mathbf{V}_3)$ $2\mathbf{V}\cdot\nabla(\mathbf{V}_2\cdot\mathbf{V}_3)$ $\omega_3\mathbf{V}_3\cdot\frac{\partial \mathbf{V}_2}{\partial p}$	$J(\omega_3, \frac{\partial \chi_3}{\partial p})$	$\omega_3\mathbf{V}_2\cdot\frac{\partial \mathbf{V}_3}{\partial p}$	5	$\oint_s\{\mathbf{V}\cdot\nabla(\mathbf{V}_2\cdot\mathbf{V}_3)+\omega_3\frac{\partial}{\partial p}(\mathbf{V}_2\cdot\mathbf{V}_3)\}\,ds$ OR $\oint_s \frac{\partial}{\partial p}\{\omega_3(\mathbf{V}_2\cdot\mathbf{V}_3)\}\,ds$	0
$2J(v_2, u_2)$	$\mathbf{V}\cdot\nabla k_2 - \xi_2\nabla\psi_2\cdot\nabla\chi_3$	$\mathbf{V}_3\cdot\nabla\xi_2$ $\xi_2 D_3$ $\omega_3\frac{\partial \xi_2}{\partial p}$ $\nabla\omega_3\cdot\nabla\frac{\partial \psi_2}{\partial p}$	$\psi_2\xi_2 D_3+\xi_2\nabla\psi_2\cdot\nabla\chi_3$ $-\psi_2\xi_2 D_3$ $-\psi_2\omega_3\frac{\partial \xi_2}{\partial p}$ $\psi_2\omega_3\frac{\partial \xi_2}{\partial p}+\omega_3\frac{\partial k_2}{\partial p}$	6	$\oint_s \mathbf{V}\cdot\nabla k_2+\omega_3\frac{\partial k_2}{\partial p}\,ds$ OR $\oint_s \frac{\partial}{\partial p}(\omega_3 k_2)\,ds$	0
$u_2\beta$ $-f\xi_2$	$-\chi_3 u_2\beta$ $\chi_3 u_2\beta - f\nabla\psi_2\cdot\nabla\chi_3$	$v_3\beta$ fD_3	$-\psi_2 v_3\beta$ $\psi_2 v_3\beta + f\nabla\psi_2\cdot\nabla\chi_3$	7	0	0
$\nabla^2\phi_1$	$\mathbf{V}\cdot\nabla\phi_1$			8	$\oint_s \mathbf{V}\cdot\nabla\phi_1\,ds$	$\frac{\partial}{\partial t}(P_1+I_1)$
$\nabla\cdot\mathbf{F}$	$\mathbf{V}_3\cdot\mathbf{F}$	$\mathbf{k}\cdot\nabla\times\mathbf{F}$	$\psi_2\cdot\mathbf{F}$	9	$\oint_s \mathbf{V}\mathbf{F}\,ds$	$\oint_M \mathbf{V}\cdot\mathbf{F}\,dM$

From combination of these two energy equations, we get—

$$\oint_s \left\{ \frac{d}{dt}(k_2+k_3+\mathbf{V}_2\cdot\mathbf{V}_3) - \frac{\partial}{\partial t}(\mathbf{V}_2\cdot\mathbf{V}_3) + \right.$$

$$\left. + \mathbf{V}\cdot\nabla\phi_1 \right\} ds = 0$$

Now, $\oint_s \mathbf{V}_2\cdot\mathbf{V}_3\, ds = \oint_s J(\psi_2, \chi_3)\, ds = 0$

$$\therefore \frac{\partial}{\partial t}\oint_s \mathbf{V}_2\cdot\mathbf{V}_3\, ds = 0$$

$$\therefore \oint_s \left\{ \frac{d}{dt}(k_2+k_3+\mathbf{V}_2\cdot\mathbf{V}_3)+\mathbf{V}\cdot\nabla\phi_1 \right\} ds = 0$$

$$i.e.,\quad \oint_s \left(\frac{d}{dt}\frac{\mathbf{V}^2}{2} + \mathbf{V}\cdot\nabla\phi_1 \right) ds = 0 \qquad (4)$$

This is the dynamically consistent energy equation which we should expect straight from the equation of motion (1) after its dot-multiplication by $\mathbf{V}$. It can be shown that if we further integrate Eq. (4) with respect to p in the vertical and use thermodynamic equation, we shall get for adiabatic frictionless flow :

$$\frac{\partial}{\partial t}\left(P_1 + I_1 + K_2 + K_3 \right) = 0 \qquad (5)$$

where,

$$P_1 = \oint_M g z\, dM$$

$$I_1 = \oint_M C_v T\, dM$$

$$K_2 = \oint_M \frac{\mathbf{V}_2^2}{2}\, dM$$

$$K_3 = \oint_M \frac{\mathbf{V}_3^2}{2}\, dM$$

$$K = \int_M \frac{\mathbf{V}^2}{2}\, dM$$

M being the mass of total atmosphere as a whole.

If energy equation (4) is satisfied in each plane, then it can be shown that energy equation (5) is necessarily satisfied over the atmospheric mass as a whole. But the converse is not true. What the truncated models do is to satisfy (5) with omission of K_3, while they satisfy (4) in more restricted forms. For example, (3,3) terms in divergence equation yield $\frac{\partial}{\partial p}(\omega_3\, k_3)$; (2,3) terms of divergence equation and (3,3) terms of vorticity equation cumulatively yield $\frac{\partial}{\partial p}\left\{ \omega_3 (\mathbf{V}_2 . \mathbf{V}_3) \right\}$; (2,2) terms in divergence equation and (2,3) terms in vorticity equation together yield $\partial/\partial p\,(\omega_3\, k_2)$. Now the vertical divergence of these vertical energy flux terms need not vanish at each horizontal level although on vertical integration w.r.t. pressure, these terms make zero contribution on assumption of $\omega_3 = 0$ at top and bottom of the atomsphere. As such, these truncated models create fictitious cancellations of vertical divergence of energy fluxes at each level and hence vertical coupling of energy in these truncated models is defective. The result of integration over entire mass of the atmosphere is also shown in Table 1.

4. We shall now examine the energetics of the various truncated models in respect of the four energy equations :

(i) from vorticity equation in a plane,
(ii) from divergence equation in a plane,
(iii) from their combination in a plane, and
(iv) from their combination over the entire mass of the atmosphere.

4.1. *Truncated Model I* — Omit $\partial D_3/\partial t$ from divergence equation, retaining all the terms of the vorticity equation.

Then, we have,

(i) $$\oint_s \left\{ \frac{\partial k_2}{\partial t} + \eta \nabla \psi_2 . \nabla \chi_3 + \omega_3 \left(\frac{\partial k_2}{\partial p} + \mathbf{V}_2 \frac{\partial \mathbf{V}_3}{\partial p} \right) \right\} ds = 0$$

(ii) $$\oint_s \left\{ -\eta \nabla \psi_2 . \nabla \chi_3 + \omega_3 \left(\frac{\partial k_3}{\partial p} + \mathbf{V}_3 . \frac{\partial \mathbf{V}_2}{\partial p} \right) + \mathbf{V} . \nabla (\mathbf{V}_2 . \mathbf{V}_3 + k_2 + k_3 + \phi_1) \right\} ds = 0$$

(iii) $$\oint_s \left\{ \left(\frac{d}{dt} \frac{\mathbf{V}^2}{2} + \mathbf{V} . \nabla \phi_1 \right) - \frac{\partial k_3}{\partial t} \right\} ds = 0$$

(iv) $$\frac{\partial}{\partial t} (P_1 + I_1 + K_2) = 0$$

4.2. *Truncated Model II* — Further, omit $u_3\beta$ and (3,3) terms from divergence equation, still retaining all the terms of the vorticity equation. Then,

(i) $$\oint_s \left\{ \frac{\partial k_2}{\partial t} + \eta \nabla \psi_2 . \nabla \chi_3 + \omega_3 \left(\frac{\partial k_2}{\partial p} + \mathbf{V}_2 . \frac{\partial \mathbf{V}_3}{\partial p} \right) \right\} ds = 0$$

(ii) $$\oint_s \left\{ -\eta \nabla \psi_2 \nabla \chi_3 + \omega_3 \left(\mathbf{V}_3 . \frac{\partial \mathbf{V}_2}{\partial p} \right) + \mathbf{V} . \nabla (\mathbf{V}_2 . \mathbf{V}_3 + k_2 + \phi_1) \right\} ds = 0$$

(iii) $$\oint_s \left[\left\{ \left(\frac{d}{dt} \frac{\mathbf{V}^2}{2} + \mathbf{V} . \nabla \phi_1 \right) - \left\{ \frac{\partial k_3}{\partial t} + \frac{\partial}{\partial p} (\omega_3\, k_3) \right\} \right] ds = 0$$

(iv) $$\frac{\partial}{\partial t} (P_1 + I_1 + K_2) = 0$$

4·3. *Truncated Model III* — Further, omit (2,3) terms from divergence equation and $J(\omega_3, \partial \chi_3/\partial p)$ terms from vorticity equation. Then the divergence equation becomes the balance equation. Now the divergence and vorticity equations are —

$$2\, J(v_2, u_2) + (u_2\beta - f\zeta_2) + \nabla^2 \phi_1 = 0$$

$$\frac{\partial \zeta_2}{\partial t} + \left(\mathbf{V}_2 . \nabla \zeta_2 + v_2\beta \right) + \left(\mathbf{V}_3 . \nabla \zeta_2 + \zeta_2 D_3 + \omega_3 \frac{\partial \zeta_2}{\partial p} + \nabla \omega_3 . \nabla \frac{\partial \psi_2}{\partial p} \right) + (v_3\beta + f D_3) = 0$$

The four energy equations become:

(i) $$\oint_s \left(\frac{\partial k_2}{\partial t} + \eta \nabla \psi_2 . \nabla \chi_3 + \omega_3 \frac{\partial k_2}{\partial p} \right) ds = 0$$

(ii) $$\oint_s \left(-\eta \nabla \psi_2 . \nabla \chi_3 + \mathbf{V} . \nabla k_2 + \mathbf{V} . \nabla \phi_1 \right) ds = 0$$

(iii) $$\oint_s \left[\left(\frac{d}{dt} \frac{\mathbf{V}^2}{2} + \mathbf{V} . \nabla \phi_1 \right) - \left\{ \frac{\partial k_3}{\partial t} + \frac{\partial}{\partial p} \left(\omega_3 k_3 + \omega_3\, \mathbf{V}_2 . \mathbf{V}_3 \right) \right\} \right] ds = 0$$

(iv) $$\frac{\partial}{\partial t} \left(P_1 + I_1 + K_2 \right) = 0.$$

4·4. *Truncated Model IV* — Further, omit $2\,J(v_2, u_2)$ terms from the divergence equation and

(2,3) terms from vorticity equation. Then the divergence and vorticity equations become—

$$u_2\beta - f\zeta_2 + \nabla^2\phi_1 = 0$$

$$\frac{\partial \zeta_2}{\partial t} + \mathbf{V}_2.\nabla\zeta_2 + v_2\beta + v_3\beta + fD_3 = 0$$

The four energy equations become:

(i) $\oint_s \left(\frac{\partial k_2}{\partial t} + f\nabla\psi_2 \cdot \nabla\chi_3\right) ds = 0$

(ii) $\oint_s \left(-f\nabla\psi_2 . \triangle\chi_3 + \mathbf{V}.\nabla\phi_1\right) ds = 0$

(iii) $\oint_s \left[\left(\frac{d}{dt}\frac{\mathbf{V}^2}{2} + \mathbf{V}.\nabla\phi_1 \right) - \left\{ \frac{\partial k_3}{\partial t} + \right.\right.$

$$\left.\left. + \frac{\partial}{\partial p}\left(\omega_3 k_3 + \omega_3 \mathbf{V}_2 . \mathbf{V}_3 + \omega_3 k_2\right) \right\} \right] ds = 0$$

(iv) $\frac{\partial}{\partial t}(P_1 + I_1 + K_2) = 0$

4.5. *Truncated Model V*—Further, neglect variation of f in dealing with expression $(u_2\beta - f\zeta_2)$ in divergence equation and $(v_3\beta + fD_3)$ in vorticity equation. Now the divergence and vorticity equations are:

$$-f_0\zeta_2 + \nabla^2\phi_1 = 0$$

$$\frac{\partial \zeta_2}{\partial t} + \mathbf{V}_2 \cdot \nabla\zeta_2 + v_2\beta + f_0 D_3 = 0$$

The four energy equations are the same as for Truncated Model IV above.

5. It is pertinent to ask in what way these defects in energetics can manifest themselves and what their cumulative effect can be; whether these are second order effects and can safely be ignored. In this respect, the following results of Sreeramamurthy (1967) suggest that the cumulative effect cannot always be ignored. He analysed linearised primitive equation model and linearised quasi-geostrophic model, using vorticity and divergence equations, in respect of baroclinic growth of unstable waves in a uniform zonal current and compared the analytical results and derived the following conclusion—

At latitude 45°N, with vertical wind shear less than 12 m $\sec^{-1}$ per 100 mb, neither the wavelength of the most unstable wave nor the magnitude of doubling time differs in the two models. At latitude 15°N, the wavelength of the most unstable wave agrees in the two models, but the doubling times are significantly different, the doubling time for the primitive equation model being about 20 to 35 per cent higher than for the quasi-geostrophic model for wind shears ranging from 2 m $\sec^{-1}$ per 100 mb to 12 m $\sec^{-1}$ per 100 mb.

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