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Wave drag by two-dimensional mountain lee waves

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सार – ऊँचाई के साथ वायु के रेखिकीय रूप से बढ़ने और स्थिरता को अपरिवर्तनीय मानते हुए द्विविमीय पर्वतीय अवरोध में स्थिर स्तरित वायु–प्रवाह वाले रेखिकीय द्रवस्थैतिक निदर्श का इस शोध–पत्र में उपयोग किया गया है। असम–बर्मा की पहाड़ियों के और भारत के पश्चिमी घाट के पर्वतीय वायुरोध और पर्वतीय अभिवाहों के विश्लेषणात्मक आँकड़े प्राप्त किए गए हैं। असम–बर्मा की पहाड़ियों के दोनों रिजों के पर्वतीय वायुरोध के सामान्य आँकड़े भी प्राप्त किए गए हैं।

ABSTRACT. A linear hydrostatic model of a stably stratified air-stream flow over a two-dimensional orographic barrier is considered assuming wind increases linearly with height and stability is constant. Analytical expressions for mountain drags and momentum fluxes are obtained for Assam-Burma hills as well as Western Ghats of India. The general expression for mountain drag also obtained for both the ridges of Assam-Burma hills.

Key words – Assam–Burma hills, Mountain drag, Momentum flux, Mountain wave and Western Ghats.

1. Introduction

When stably stratified flow across an orographic barrier, gravity waves are generated and propagate upwards, transferring horizontal momentum vertically. Because of the orographic waves the pressure is systematically higher on the upwind slopes than the downward slopes and thus exerting a net force on the ground. This force is known as mountain drag. This drag is an important part of the atmospheric momentum balance various theoretical studies are made of stably stratified airflow across orographic barrier related to mountain drag like Queney (1947 & 1948), Blumen (1965), Bretherton (1969), Lilly (1972), Smith (1978) etc.

Mountain wave problem addressing properties of mountain waves over Indian region was studied by many authors like Das (1964), Sarker (1965, 1966 & 1967), Sarker *et al*. (1978), Kumar *et al*. (1998), Kumar (2000), Dutta (2001), Dutta *et al.* (2002) etc. Kumar *et al*. (1998) developed a 2-D analytical model for real time prediction of mountain wave due to Pirpanjal mountains over Kashmir valley. He presented the solution for vertical velocity and displacement for all values of wave numbers. Kumar (2000) prepared the climatology of mountain waves over Indian region based on the NOAA satellite imageries. He showed that Lee waves occur in the wavelength spectrum of 8 to 18 km over Indian sub-continents. Dutta (2001) obtained analytical expression for mountain drag, momentum flux and energy flux across 2-D profile of Western Ghats. Very recently Dutta and Naresh (2005) studied fluxes of momentum and energy generated by mountain waves across Assam-Burma hills of India. They showed the impact of valley between the ridges of Assam `- Burma hills of India for generation of mountain drag and energy flux. But all studies related to mountain drag and energy flux across Indian orographic barriers based on fact that wind is constant with height.

Aim of the present study is to develop a mathematical model to obtain the analytical expressions for mountain drag and momentum flux for wind, which increases linearly with height over Western Ghats as well as Assam- Burma hills of India.

2. The mathematical model

We consider a steady, frictionless, adiabatic flow of a vertically unbounded stratified, inviscid and Boussinesq fluid across a two-dimensional orography. It is assumed that basic wind flow linearly increasing with height, $U = U_0(1+cz)$, where c is the shear parameter and U_0 is the surface wind speed. The horizontal dimension of the hills as well as the disturbance is taken to be small enough so that the effect of Coriolis force may be neglected. We further assume that flow is independent of y coordinate. Under the above assumption, the linearised governing equations may be written as :

$$\rho_0 U \frac{\partial u'}{\partial x} + \frac{\partial p'}{\partial x} + \rho_0 w' \frac{\partial U}{\partial z} = 0 \qquad (1)$$

$$\rho_0 U \frac{\partial w'}{\partial x} + \frac{\partial p'}{\partial z} + \rho' \mathrm{g} = 0 \qquad (2)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad (3)$$

$$U \frac{\partial \rho'}{\partial x} + \frac{\mathrm{d}\bar{\rho}}{\mathrm{dz}} w' = 0 \qquad (4)$$

where u', w', p' and ρ' are respectively the perturbation zonal wind, vertical wind, pressure and density. The mean density ρ_0, gravitational acceleration g and density gradient $\frac{\mathrm{d}\bar{\rho}}{\mathrm{dz}}$ are taken as constant. The Brunt-Vaisala frequency $N^2 = -\frac{\mathrm{g}}{\rho_0}\frac{\mathrm{d}\bar{\rho}}{\mathrm{dz}}$ is assumed to be constant.

Near the ground the vertical velocity must satisfy the boundary condition

$$w'(x, z=0) = U_0 \frac{\partial h}{\partial x} \qquad (5)$$

where $h(x)$ is the profile of orographic barrier.

Now, if $\hat{f}(k,z)$ be the Fourier transform of function $f(x,z)$, then they are related by

$$\hat{f}(k,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,z)\exp(-ikx)\mathrm{d}x \qquad (6A)$$

$$f(x,z) = \int_{-\infty}^{\infty} \hat{f}(k,z)\exp(ikx)\mathrm{d}k \qquad (6B)$$

Now performing Fourier transforms to Eqns. (1) to (4), we obtained

$$ik\rho_0 U\hat{u} + ik\hat{p} + \rho_0 \hat{w}\frac{\partial U}{\partial z} = 0 \qquad (7)$$

$$ik\rho_0 U\hat{w} + \frac{\partial \hat{p}}{\partial z} + \hat{\rho}\mathrm{g} = 0 \qquad (8)$$

$$ik\hat{u} + \frac{\partial \hat{w}}{\partial z} = 0 \qquad (9)$$

$$ikU\hat{\rho} + \frac{\mathrm{d}\bar{\rho}}{\mathrm{dz}}\hat{w} = 0 \qquad (10)$$

where, $\hat{u}$, $\hat{w}$, $\hat{\rho}$ are respectively the Fourier transform of u', w', ρ'.

The above system of equations from (7) to (10) after simplification reduces to

$$\frac{\partial^2 \hat{w}}{\partial z^2} - \left(k^2 - \frac{N^2}{U^2} + \frac{1}{U}\frac{\partial^2 U}{\partial z^2}\right)\hat{w} = 0 \qquad (11)$$

Now assuming hydrostatic approximation, which is equivalent to assuming the k^2 term is negligibly small in Eqn. (11), hence

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \left(\frac{N^2}{U^2} - \frac{1}{U}\frac{\mathrm{d}^2 U}{\mathrm{d}z^2}\right)\hat{w} = 0 \qquad (12)$$

Substituting $U = U_0(1+cz)$ and $Z = 1+cz$ into Eqn. (12) and simplifying, we get

$$\frac{\partial^2 \hat{w}}{\partial Z^2} + \frac{R_i}{Z^2}\hat{w} = 0 \qquad (13)$$

where, R_i^2 is the Richardson number and defined by $R_i^2 = \frac{N^2}{(U_0 c)^2}$ and considered as a constant.

The solution of Eqn. (13) is

$$\hat{w}(k,z) = \mathrm{A}Z^{1/2}e^{i\lambda \ln Z} + \mathrm{B}Z^{1/2}e^{-i\lambda \ln Z} \qquad (14)$$

where, $\lambda = \sqrt{\left(R_i - \frac{1}{4}\right)}$

As energy is propagated upward, so B should be equal to zero in Eqn. (14)

$$\hat{w}(k,z) = \mathrm{A}Z^{1/2}e^{i\lambda \ln Z} \qquad (15)$$

Now by Fourier transform of Eqn. (5) and using into Eqn. (15), we get

$$\hat{w}(k,z) = ikU_0\hat{h}(k)Z^{1/2}e^{i\lambda \ln Z} \tag{16}$$

here, $\hat{w}(k,z)$ is the Fourier form of perturbation vertical velocity.

3. Mountain drag

As the total drag of the air on the mountain is obtained by integrating the momentum transport along the x - axis. For steady state wave drag is certainly associated with the downstream pressure drop across the topography and thus a force on the earth. This force, per unit length of topography is just equal to

$$\mathrm{D}_{z=0} = -\rho_0 \int_{-\infty}^{\infty} u'w'\mathrm{d}x$$

where $\mathrm{D}_{z=0}$ is the drag at the mountain which is just negative of the Reynolds stress evaluated on the surface $z = 0$

Now the Mountain drag by Gill (1982) at any level is

$$\mathrm{F} = \int_{-\infty}^{\infty} p'\mathrm{d}x = \int_{-\infty}^{\infty} p'\frac{\mathrm{d}\eta'}{\mathrm{d}x}\mathrm{d}x = -\int_{-\infty}^{\infty} \eta'\frac{\mathrm{d}p'}{\mathrm{d}x}\mathrm{d}x \tag{17}$$

here, perturbation are assumed to be zero at $x = \infty$ or $x = -\infty$ and $\eta'(x,z)$ is the height of the streamline above undisturbed level.

Using Paraseval's theorem for Fourier integral, The Mountain drag becomes

$$F = -2\pi i \int_{-\infty}^{\infty} k\hat{p}\hat{\eta}^*\mathrm{d}k \tag{18}$$

where, $\hat{p}^*$ is complex conjugates of $\hat{p}$.

As, $w' = U\dfrac{\partial\eta'}{\partial x}$

therefore, by its Fourier transform, we get

$$\hat{w} = ikU\hat{\eta} \tag{19}$$

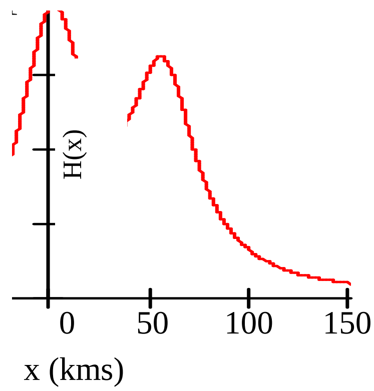

Fig. 1. 2- D profile of Assam- Burma hills

Now, using Eqns. (7), (9) and (19) into Eqn. (18), we have

$$F = -2\pi\rho_0 \Re\left\{\int_{-\infty}^{\infty} i\frac{1}{k}\left(\frac{\partial\hat{w}}{\partial z} - \hat{w}\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}z}\right)\hat{w}^*\mathrm{d}k\right\} \tag{20}$$

Finally, substitute Eqn. (16) into Eqn. (21), we get

$$F = -2\pi c\rho_0 U_0{}^2 \Re\left\{\int_{-\infty}^{\infty} i\left[-\frac{1}{2} + i\lambda\right]k\hat{h}(k)\hat{h}^*(k)\mathrm{d}k\right\} \tag{21}$$

As momentum flux is equal to negative of mountain drag (Dutta, 2001), therefore

Momentum flux is written as

$$M = 2\pi c\rho_0 U_0{}^2 \Re\left\{\int_{-\infty}^{\infty} i\left[-\frac{1}{2} + i\lambda\right]k\hat{h}(k)\hat{h}^*(k)\mathrm{d}k\right\} \tag{22}$$

4. Mountain drag along Assam-Burma hills

Now a 2-D profile of Assam-Burma hills as shown in Fig. 1 has been considered to solve Eqn. (22), whose analytical expression considered by De (1973) is

$$h(x) = \frac{a^2 b_1}{a^2 + x^2} + \frac{a^2 b_2}{a^2 + (x - \mathrm{D})^2} \tag{23}$$

where,

$a = 20.0$ km, $b_1 = 0.9$ km, $b_2 = 0.7$ km and

$\mathrm{D} = 55.0$ km

By Fourier transform of Eqn. (23), we get

$$\hat{h}(k) = ae^{-ak}\left(b_1 + b_2 e^{-iDk}\right) \quad (24)$$

Put Eqn. (24) into Eqn. (22) for real solution, we get

$$F_A = -2\pi c \rho_0 U_0{}^2 \Re \left\{ i\left[-\frac{1}{2} + i\lambda\right] \int_{-\infty}^{\infty} k\left(b_1{}^2 + b_2{}^2 + 2b_1 b_2 \cos Dk\right) e^{-2ak} \mathrm{d}k \right\}$$

$$= 2\pi\rho_0 a^2 NU_0 \left(1 - \frac{1}{4R_i}\right)^{1/2} \left[\left(b_1{}^2 + b_2{}^2\right)\frac{1}{4a^2} + 2b_1 b_2 \frac{\left(4a^2 - D^2\right)}{\left(4a^2 + D^2\right)^2}\right] \quad (25)$$

So, momentum flux becomes

$$M_A = -2\pi\rho_0 a^2 NU_0 \left(1 - \frac{1}{4R_i}\right)^{1/2} \left[\left(b_1{}^2 + b_2{}^2\right)\frac{1}{4a^2} + 2b_1 b_2 \frac{\left(4a^2 - D^2\right)}{\left(4a^2 + D^2\right)^2}\right] \quad (26)$$

The mountain drag due to first ridge (*i.e.*, $\frac{a^2 b_1}{a^2 + x^2}$) is

$$F_{1A} == \frac{1}{2}\pi\rho_0 aNU_0 b_1{}^2 \left(1 - \frac{1}{4R_i}\right)^{1/2} \quad (27)$$

and mountain drag due to second ridge (*i.e.*, $\frac{a^2 b_2}{a^2 + (x-D)^2}$) is

$$F_{2A} == \frac{1}{2}\pi\rho_0 aNU_0 b_1{}^2 \left(1 - \frac{1}{4R_i}\right)^{1/2} \quad (28)$$

From Eqns. (25), (27) and (28), it is clear that

$$F_A \neq F_{1A} + F_{2A}$$

which shows that mountain drag due to entire Assam-Burma hills is not equal to the sum of the mountain drags due to two ridges, this indicate that valley between the two ridges of Assam-Burma hills also contribute for generation of mountain drag.

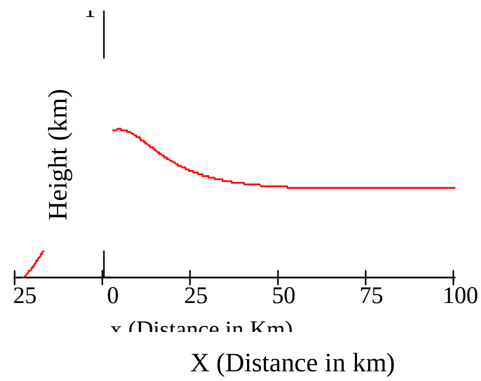

Fig. 2. 2- D profile of Western Ghats

5. Mountain drag along Western Ghats

Now a 2-D profile of Western Ghats as shown in Fig. 2 has been considered to find mountain drag, whose analytical expression considered by Sarker (1965) is

$$h(x) = \frac{H}{1 + \frac{x^2}{a^2}} + b \tan^{-1}\frac{x}{a} \quad (29)$$

where, $a = 18.0\text{km}$, $H = .52\text{km}$, $b = \frac{2}{\pi} \times .35\text{km}$.

By Fourier transform of Eqn. (29) is

$$\hat{h}(k) = \left[aH - i\frac{b}{k}\right]e^{-ak} \quad (30)$$

Substitute Eqn. (30) into Eqn. (22) for real solution, we get

$$F_W = \frac{1}{2}\pi\rho_0 NU_0 H^2 \left(1 - \frac{1}{4R_i}\right)^{1/2} \quad (31)$$

Similarly momentum flux for Western Ghats becomes

$$M_W = \frac{1}{2}\pi\rho_0 NU_0 H^2 \left(1 - \frac{1}{4R_i}\right)^{1/2} \quad (32)$$

6. Results and discussions

In the course of analysis of this paper, we summarize the results as follows :

Using the profile of Western Ghats due to Sarker (1965) and Assam- Burma hill of India due to De (1973), we have derived the analytical expressions for mountain drags and momentum fluxes for entire Assam-Burma hills (Eqn. 25) and both the ridges (Eqns. 27 & 28) of the Assam-Burma hills with wind increasing linearly with height. Analytical expression for mountain drag and momentum flux for Western Ghat (Eqns. 31 & 32) has been also derived.

If the wind variation is very slow with height *i.e.*, $c << 1 \Rightarrow R_i >> 1$. So the Eqn. (25) gives us the following result

$$(F_A)_{R_i>>1} = F_{A0} = 2\pi\rho_0 a^2 NU_0 \left[\left(b_1^2 + b_2^2\right)\frac{1}{4a^2} + 2b_1 b_2 \frac{\left(4a^2 - D^2\right)}{\left(4a^2 + D^2\right)^2} \right]^2$$

Which agree with the mountain drag obtains by Dutta and Naresh (2005).

Similarly for $R_i >> 1$ Eqn. (31) becomes

$$(F_W)_{R_i>>1} = F_{W0} = \frac{1}{2}\pi\rho_0 NU_0 H^2$$

This result agrees with the result due to Dutta (2001). Now substitute F_{A0} and F_{W0} into Eqn. (25) and Eqn. (31) respectively, we have

$$\frac{F_A}{F_{A0}} = \frac{F_W}{F_{W0}} = \left(1 - \frac{1}{4R_i}\right)^{1/2}$$

that is, to say that above normalized mountain drag is the function of $1/R_i$ and is independent on the orographic barrier. Its variation with $1/R_i$ is shown in Fig. 3.

Above Fig. 3. shows that as $1/R_i$ increases, normalized mountain drag decrease and approaches zero, when $1/R_i \rightarrow 1/4$.

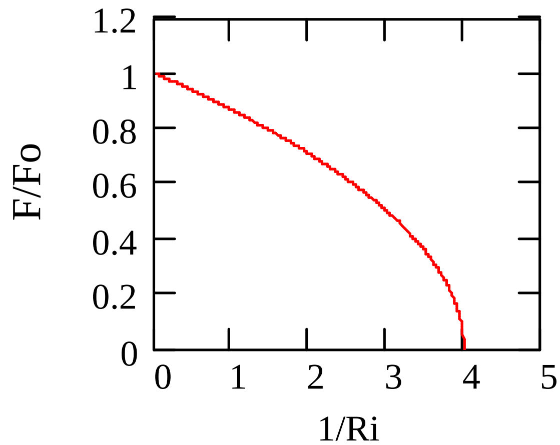

Fig. 3. Variation of mountain drag

In Eqns. (31) and (32), a factor '*b*' for plateau part is does not appear, so we may say that plateau part of the western ghat does not contribute towards the generation of the mountain drag and momentum flux, which is conformity with the earlier findings of Dutta (2001).

As factor '*a*' is absent in Eqns. (31) and (32), which show that drag and flux are independent on the half width of the bell shaped portion of Western Ghats. Which is not true in case of Assam-Burma hills, this may be due to the valley between the two ridges of hills.

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