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Analytical methods for calculating precipitating relativistic electron effects in the D region

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ABSTRACT. Equations for calculating the penetration and loss of energy by precipitating relativistic electrons in the D region of the atmosphere, as well as the amount of ionization caused by these electrons are derived and discussed beginning with basic physical principles related to high energy electrons.

1. Introduction

Part of the ionization of the D region between 70 and 90 km occurs as a result of the penetration of precipitating high energy, relativistic electrons to those altitudes, ionizing the air molecules by collision. At higher altitudes, relativistic electrons are believed to be responsible for excitation processes such as airglow and auroral emissions, [λ 5577 Å by O (^{1}S), λ 3914 Å etc by $N_2^+(B^2\,\Sigma u^+)$, (Walker 1972), as well as for part of the ionization in that region. Precipitating energetic electrons are also believed to be an important source for the maintenance of ionization in the D region at night (Potemra and Zmuda 1970, Radicella 1968).

From data obtained by O'Brien (1964) the average flux of precipitating electrons varies from 10^2 to 10^3 particle cm^{-2} sec^{-1} sterad^{-1} at mid-latitudes to approximately 10^5 particle cm^{-2} sec^{-1} sterad^{-1} around the auroral zone (above the atmosphere). In that study it was also found that the auroral fluxes varied over a range of a factor of 10^4, hence the figures quoted as average values tend to mask the variable nature of this phenomenon. Fluxes and energy spectrum distributions of precipitating electrons are basically very time dependent.

Analytical methods for calculating the penetration of precipitating relativistic electrons to various altitudes within the D region and their loss of energy due to collisions with air molecules as well as methods for calculating the extent of the ionization to be expected from an assumed precipitating electron energy spectrum are important for gaining insight into the extent to which these energetic electrons contribute to the ionization of the D region. This note is concerned with the physics as well as the mathematical steps involved in deriving the equations required for this insight.

2. Derivation of the energy loss equation for relativistic electrons in the D region

The fundamental equation for the energy loss, per unit path length, for high energy relativistic electrons in passing through matter is given by Bethe and Ashkin (1953) [equation (52), p. 254)] as

$$-dE/dx = 2\cdot\pi\cdot(N\cdot e^4\cdot Z/(m_0\cdot v^2))\cdot (\ln(m_0\cdot v^2\cdot E)/(2\cdot I^2\cdot(1-\beta^2))) - (2\cdot(1-\beta^2)^{1/2} - 1+\beta^2)\cdot\ln(2) + 1-\beta^2 + (1-(1-\beta^2)^{\frac{1}{2}})^2/8) \qquad (2\cdot1)$$

where,

E is the kinetic energy of the incident electron.

x is the distance along the path through the material.

N is the number of atoms of the material per unit volume.

e is the magnitude of the charge of an electron.

Z is the nuclear charge of an atom of the material.

m_0 is the mass of an electron within an atom (effectively, the rest mass of an electron).

v is the velocity of the incident electron.

I is the average excitation potential over *all* electrons of an atom.

$\beta \equiv v/c$ and c is the velocity of light.

Equation (2·1) is general and also applies to low energy non-relativistic electrons, as well. A more convenient form of eq. (2·1), useful for working within the limited range of electron energies of precipitating energetic electrons in the atmosphere, is

$$-dE/dx = 2\cdot\pi\cdot(N\cdot e^4\cdot Z/(m_0\cdot v^2))\cdot (\ln((m_0\cdot v^2\cdot E)/2\cdot I^2\cdot(1-\beta^2))-C'') \quad (2\cdot 2)$$

where C'' is a constant.

By multiplying and dividing by the actual mass of the incident electron, m, eq. (2·2) becomes

$$-dE/dx=((\pi\cdot e^4)\cdot(NZ)\cdot(m/m_0)/(m\cdot v^2/2))\cdot \ln(E\cdot(m\cdot v^2/2)\cdot(m_0/m)/(C'I^2(1-\beta^2))) \quad (2\cdot 3)$$

where C' is a constant.

Since the ratio of the relativistic mass to the rest mass, m/m_0, is $(1-\beta)^{-1/2}$, the latter quantity may be substituted for the ratio in the equation. Also, at lower relativistic velocities, such as those possessed by most of the precipitating electrons in the ionosphere, a good approximation for the kinetic energy is $m\cdot v^2/2$, for which E may be substituted.

Hence, eq. (2·3) may be changed to

$$-dE/dx=(\pi\cdot e^4\cdot N\cdot Z/(E.(1-\beta^2)^{\frac{1}{2}}))\cdot \ln(E^2/(C'\cdot I^2\cdot(1-\beta^2)^{\frac{1}{2}})) \quad (2\cdot 4)$$

The expression $(1-\beta^2)^{\frac{1}{2}}$ does not vary sufficiently to alter eq. (2·4) significantly, over the limited range of values for the kinetic energy applicable to the D region, hence the following may be assumed as constants in eq. (2·4):

$$a=\pi\cdot e^4/(1-\beta^2)^{\frac{1}{2}}$$

$$b=1/(C'\cdot I^2\cdot(1-\beta^2)^{\frac{1}{2}})$$

Substituting a and b into eq. (2·4) yields

$$-dE/dx=(a\cdot N\cdot Z/E)\cdot\ln(b\cdot E^2) \quad (2\cdot 5)$$

The equation obtained by multiplying both members of eq. (2·5) by $-(2\cdot b\cdot E\cdot dx)/\ln(b\cdot E^2)$ and then integrating is

$$\int_{E_0}^{E_f} d(b\cdot E^2)/\ln(b\cdot E^2) = -2.a\cdot b\cdot\int N\cdot Z\cdot dx \quad (2\cdot 6)$$

Therefore,

$$E_i(2\cdot\ln(b^{\frac{1}{2}}\cdot E_f)) = E_i(2\cdot\ln(b^{\frac{1}{2}}\cdot E_0)) -2\cdot a\cdot b\cdot\int Z\cdot N\cdot dx \quad (2\cdot 7)$$

where, E_i is the exponential integral function.

E_f is the electron energy after passing into a region of the atmosphere.

E_0 is the electron energy at the top of the atmosphere.

When a high energy charged particle is slowed down to velocities corresponding to the thermal energy of the material through which it passes, it may be considered as being stopped (Bethe and Ashkin 1953, p.166). Substituting this final energy for E_f makes the left-hand member of eq. (2·7) effectively zero, thus

$$\int Z\cdot N\cdot dx=1(/(2\cdot a\cdot b))\cdot E_i(2\cdot\ln(b^{\frac{1}{2}}\cdot E_0)) \quad (2\cdot 8)$$

The kinetic energy of a relativistic electron is given by

$$E=m\cdot c^2-m_0\cdot c^2$$

and since the ratio of the relativistic mass to the rest mass, m/m_0, is $(1-\beta^2)^{-\frac{1}{2}}$,

$$(1-\beta^2)^{-1/2}=1+E/(m_0\cdot c^2) \quad (2\cdot 9)$$

The distance through which a charged, high energy particle penetrates a given material before being stopped is known as the range of the particle for that material. This may be determined for relativistic electrons penetrating air under a given set of conditions of temperature and pressure. For this case, N and Z are constant in eq. (2·8) and the integration yields the range of the relativistic electron as

$$R=(1/(2\cdot a\cdot b\cdot Z\cdot N))\cdot E_i(2\cdot\ln(b^{\frac{1}{2}}\cdot E_0)) \quad (2\cdot 10)$$

A graph of the range-energy relation for high energy electrons in air at 0°C and 1 atmosphere pressure is shown in Fig. 1. The agreement with experimental values cited by Landolt-Börnstein (1952) and by Omholt (1965) is quite good in the interval between 10 kev and 200 kev and for values in the vicinity of this interval of energy. Most of the precipitating relativistic electrons penetrating into the D region have energy values of approximately this magnitude. Electrons with smaller energy values are stopped before reaching the D region.

Generally, precipitating electrons follow helical type of paths about the geomagnetic field lines of the earth. The derived equations should be modified to include the effect of the pitch angle, θ, of the electron path with respect to the field lines. If the geomagnetic field lines are assumed to be approximately vertical in high-latitude regions and if y is used to represent the vertical component of the actual distance traversed, the energy-loss equation for relativistic electrons in the D region becomes

$$-dE/dy=(a\cdot N\cdot Z/E)\cdot\ln(b\cdot E^2)\cdot\sec\theta \quad (2\cdot 5a)$$

When equations (2·7) and (2·8) are changed to include the effect of the pitch angle, these equations become, respectivley,

$$E_i\,(2\cdot\ln(b^{\frac{1}{2}}\cdot E_f)\,)=E_i(2\cdot\ln(b^{\frac{1}{2}}\cdot E_0)\,)\;-$$

$$-2\cdot a\cdot b\cdot\int Z\cdot N\cdot(\sec\theta)\cdot dy \qquad (2\cdot 7a)$$

$$\int Z\cdot N\cdot(\sec\theta)\cdot dy=(1/(2\cdot a\cdot b)\,)\cdot E_i(2\cdot \ln\,(b^{\frac{1}{2}}\cdot E_0)\,) \qquad (2\cdot 8a)$$

For the region below about 100 km it is a good approximation to assume Z to be a constant value, so that this quantity may be taken outside of the integral sign in eq. (2·8a). Furthermore, for the purpose of calculating penetration depth into the region below about 100 km and for calculating ionization rates in the D region, the pitch angle, θ, may be regarded as approximately constant over a limited interval and hence also be taken outside of the integral sign (the quantity $E\cdot\sin^2\theta/B$ remains constant, where B is the geomagnetic field strength—Omholt 1965). Therefore, by dividing eq. (2·8a) by eq. (2·10) and assuming the same kinetic energy in each equation, the column density is found to be

$$\eta=\tfrac{1}{2}\int N\cdot dy = N_{STP}\cdot R\cdot\cos\theta/2 \qquad (2\cdot 11)$$

where N_{STP} is very closely twice the Loschmidt's number, $2\cdot 68719\cdot 10^{19}$ cm^{-3} (Weast 1970, p. F-84).

The range of the energetic electron, given by eq. (2·10), may be approximated by

$$R=4\cdot 467.10^{-3}.\mathrm{cm}.(\mathrm{kev})^{-1\cdot 7285}.E^{1\cdot 7285} \quad (2\cdot 12)$$

where the kinetic energy E, is expressed in kev units.

If the range, R, as given in eq. (2·12) is substituted into eq. (2·11), and then if the resulting equation is solved for E, the following expression is obtained:

$$E=(\eta/(1\cdot 20.10^{17}.\mathrm{cm}^{-2})\,)^{0\cdot 5785}\cdot\mathrm{kev}\cdot(\sec\theta)^{0\cdot 5785} \quad (2\cdot 13)$$

Since the column density, η, is determined at a given altitude, eq. (2·13) yields the minimum energy required by a relativistic electron to reach that altitude, as a function of the pitch angle, θ.

3. Derivation of the ionization rate equation

The ionization rate, q, at a given altitude will be obtained only from those electrons of an assumed energy spectrum distribution having sufficient energy to penetrate to the altitude under consideration. Those electrons of the assumed distribution having less than the required energy will be stopped at higher altitudes in accordance with eq. (2·8a) or eq. (2·13). In addition, each high energy electron creates ion-pairs along its entire path length before it is stopped. Hence, the number of ion-pairs created at a given altitude

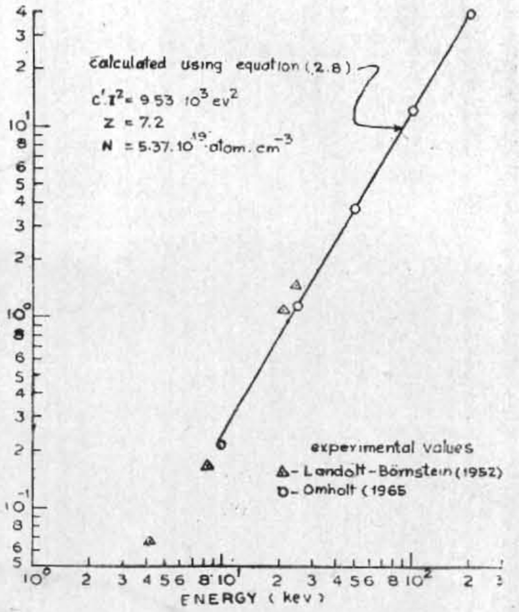

Fig. 1

Range-energy relation for electrons in air at 0°C and 760 mm Hg.

will be the sum of the rate of energy loss per unit length by each electron in the entire portion of the electron energy spectrum (at each differential solid angle, $2\cdot\pi.\sin\theta d\theta$ sterad), having sufficient energy to penetrate to the altitude under consideration, divided by the average energy loss per ion-pair created, $(\triangle E)_{av}$.

If $J\,(E)$ refers to the function for the assumed energy spectrum distribution of precipitating electron flux, then the number of ion-pairs created at a given altitude is given by

$$q=\int_0^{\theta m}\int_{El\,(\theta)}^{Em} 2.\;\pi.\;\sin\theta\cdot d\theta.\;\mathrm{sterad}.\,(-dE/dy).\;.J(E)\cdot dE/(\triangle E)_{av} \qquad (3\cdot 1)$$

where,

- q is the ionization rate (ion-pair. cm^{-3}. sec^{-1}).
- θ is the angle with respect to the vertical (the pitch-angle of precipitating electrons at a given solid angle).
- θ_m is the maximum possible pitch-angle for the electrons with the largest energy of the distribution, (by means of eq. (2·13).
- E_l is the least energy of the distribution that an electron may have to penetrate to the given altitude at a given pitch angle, θ.
- E_m is the maximum energy of the assumed electron energy distribution.

Assuming isotropy for the electron flux over the upper hemisphere and a function for the electron energy spectrum distribution similiar to the model form used by Potemra and Zmuda (1970).

$$J(E) = K \cdot E^{-\gamma} \qquad (3\cdot2)$$

where K and γ are constants, and by substituting the expression for the energy loss per unit vertical distance, eq. (2·5a), into eq. (3·1), this equation becomes

$$q = \alpha \cdot \int_0^{\theta_m} \tan\theta \cdot d\theta \cdot \int_{E_l(\theta)}^{E_m} E^{-(1+\gamma)} \cdot \ln(b \cdot E^2) \cdot dE \qquad (3\cdot3)$$

where $\alpha = 2 \cdot \pi \cdot \text{sterad.}\ a.\ N.\ Z.\ K/(\triangle E)_{av}$.

By multiplying and dividing by $2 \cdot E . b^{(1+\gamma/2)}$, eq. (3·3) is transformed into the following expression

$$q = (\alpha \cdot b^{\gamma/2}/2) \int_0^{\theta_m} \tan\theta \cdot d\theta \cdot \int_{E_{l(\theta)}}^{E_m} \frac{(\ln(b \cdot E^2) . d(b \cdot E^2))}{(b \cdot E^2)^{(1+\gamma/2)}} \qquad (3\cdot4)$$

The integration of eq. (3·4) with respect to E, without substituting the limits of integration, yields

$$q = (\alpha \cdot b^{\gamma/2}/2) . \int_0^{\theta_m} \tan\theta \cdot d\theta \cdot (b \cdot E^2)^{-\gamma/2} \cdot (-(2/\gamma) . \cdot \ln(b \cdot E^2) - (2/\gamma)^2) \qquad (3\cdot5)$$

Multiplying and dividing by $b^{-\gamma/2}$ and substituting the upper and lower limits of integration for E changes eq. (3·5) into an expression with only the energetic electrons' pitch angle, θ, as the variable to be integrated [since E_m is a constant value and $E_l(\theta)$ is a function of θ], thus

$$q = (2 \cdot \alpha/\gamma^2) \cdot \int_0^{\theta_m} \tan\theta \cdot d\theta \cdot ((1 + (\gamma/2) \cdot \ln(b \cdot E_l(\theta)^2))/(E_l(\theta))^{\gamma} - (1 + (\gamma/2) \cdot \ln(b \cdot E_m^2))/E_m^{\gamma}) \qquad (3\cdot6)$$

Eq. (3·6) may now be subdivided into the following terms:

$$q = (2 \cdot \alpha/\gamma^2) . ((1 + (\gamma/2) \cdot \ln(b)) \cdot \int_0^{\theta_m} \tan\theta \cdot d\theta / (E_l(\theta))^{\gamma} + \gamma . \int_0^{\theta_m} \ln(E_l(\theta)) \cdot \tan\theta \cdot d\theta / (E_l(\theta))^{\gamma}$$
$$+ ((1 + (\gamma/2) \cdot \ln(b \cdot E_m^2)) / E_m^{\gamma}) \cdot \ln(\cos\theta_m)) \qquad (3\cdot7)$$

It is clear that a precipitating electron penetrating to a given altitude with a pitch angle, θ, requires a larger energy than an electron penetrating to the same altitude whose pitch-angle is zero, since the latter encounters the fewest collisions with air molecules. For precipitating electrons in the D region, a good approximation to the dependence of the minimum kinetic energy required, as a function of θ, may be obtained by the use of eq. (2·13). For the purpose of using this relation in this derivation, a symbolic form for eq. (2·13) can be

$$E_l(\theta) = E' \cdot (\sec\theta)^p \qquad (3\cdot8)$$

Substituting for $E_l(\theta)$ and multiplying and dividing by $\sec\theta$ in eq. (3·7) changes this expression to

$$q = (2 \cdot \alpha / (\gamma^2 \cdot E'^{\gamma})) \cdot (\gamma (1/\gamma . + \ln(b^{1/2} \cdot E')) \cdot \int_0^{\theta_m} (\sec\theta)^{-(1+p\gamma)} \cdot d(\sec\theta) + p . \gamma . \int_0^{\theta_m} (\sec\theta)^{-(1+p\gamma)} .$$
$$\cdot \ln(\sec\theta) \cdot d(\sec\theta) + (E'/E_m)^{\gamma} \cdot (1 + \gamma \cdot \ln(b^{1/2} \cdot E_m)) \cdot \ln(\cos\theta_m)) \qquad (3\cdot9)$$

Completing the integration of the first two terms in the parenthesis of the right-hand member of eq. (3·9) over the angle, θ, from $\theta = 0$ to $\theta = \theta_m$ (for the electrons with the maximum kinetic energy), finally yields the ionization rate equation for precipitating electrons in the D region

$$q = (2 \cdot \alpha / (\gamma^2 \cdot E'^{\gamma} \cdot p)) \cdot (A_1 + A_2 + A_3) \qquad (3\cdot10)$$

where,

$$A_1 = (1/\gamma + \ln(b^{1/2} \cdot E')) \cdot (1 - (\cos\theta_m)^{p\gamma}) \qquad (3\cdot10a)$$

$$A_2 = (1 + (\gamma \cdot \ln (\cos \theta_m)^p - 1) \cdot (\cos \theta_m)^{p\gamma}) / \gamma \quad (3\cdot10b)$$

$$A_3 = (1 + \gamma \cdot \ln (b^{1/2} \cdot E_m)) \cdot (\cos \theta_m)^{p\gamma} \cdot \ln (\cos \theta_m)^p \quad (3\cdot10c)$$

$$E' = (\eta / (1\cdot20.10^{17} . \text{cm}^{-2}))^{0\cdot5785} \cdot \text{kev} \quad (3\cdot10d)$$

$$p = 0\cdot5785 \quad (3\cdot10e)$$

$$(\cos \theta_m)^p = E'/E_m \quad (3\cdot10f)$$

The meaning of the other symbols are given in Sections 2 and 3 of this paper. An average energy loss per ion-pair created, $(\triangle E)_{av}$, of about 35· ev. has found in laboratory work, for air.

4. Discussion

In deriving the equations of Sections 2 and 3 of this paper, it was assumed that the primary electron losses were due entirely to ionization and excitation losses during collision with the atoms and molecules of the atmosphere and additional effects were neglected. At very high energies, another type of collisional loss is the bremsstrahlung energy loss, a radiation loss [*e.g.*, X-rays in aurora (Omholt 1965a)]. The following equation (Bethe and Ashkin 1953), may be used to estimate the extent of this type of energy loss to determine whether or not this is large enough to justify additional calculation :

$$-(dE/dx)_{rad} \approx (Z \cdot E / (1600 \cdot (m_0c^2)) \cdot (-(dE/dx)_{coll}) \quad (4\cdot1)$$

where,

$-(dE/dx)_{rad}$ is the bremsstrahlung energy loss.
$-(dE/dx)_{coll}$ is the ionization and excitation loss.

For the D region, eq. (4·1) shows that for precipitating relativistic electrons with energies less than 1. mev the bremsstrahlung loss will be less than 1% of the ionization and excitation loss. Eq. (2·7a) may be employed to evaluate the magnitude of the remaining energy of an electron penetrating a given altitude, which can be used to determine the extent of the bremsstrahlung energy loss.

The value of Z for the atmosphere in all of the equations is the weighted average value of the nuclear charge of each type of atom in the mixture of atmospheric gases at each altitude. According to the Cira (1965) model atmosphere tables, below an altitude of about 100 km the proportions of the mixture of gases are very nearly constant, thus Z may be regarded as a constant. Above about 100 km the proportions of the mixture of atmospheric gases vary with altitude, so that Z is no longer constant in that region.

From high-latitude observations with the satellite Injun 3 (O'Brien 1964), it was concluded that the angular distribution of precipitating electron flux tends towards isotropy over the upper hemisphere. This observation was utilized during the derivation of the ionization rate equation. If this were not so then the variation of electron flux with the solid angle would have had to be included in the derivation when the electron energy spectrum distribution function, $J(E)$, is used.

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