Indian J. Met. Geophys. (1974), 25, 1, 39-45

551.558.21

Air pocket theory for large amplitude waves in stratified fluids

B. K. SEN GUPTA

Portsmouth Polytechnic, England

(*Received 19 September 1972*)

ABSTRACT. The general wave equation for steady-state two-dimensional flow in stratified fluids has been re-written in an integral form. (This was first suggested by Long and later exploited by Yih). An Airy integral solution is presented in this paper, and the possibility of obtaining the widest range of exact solutions examined.

1. Notation

The important notation is set out below. Suffix 0 has been used to refer to Lagrangian quantities, and all such will be constant along a streamline.

x, z : space co-ordinates in two-dimensional system

δ : vertical displacement from the undisturbed position

z_0 : height of streamline when undisturbed

U_0 : undisturbed stream velocity

ρ_0 : density

β_0 : static stability $= -\frac{1}{\rho_0} \cdot \partial\rho_0/\partial z_0$

k : horizontal wave number

$L_0^2 = G_0$: stability parameter $= g\beta_0/U_0^2$

$R_0 : \rho_0^{\frac{1}{2}} \; U_0$

$$\eta_0 : \int_{z_1}^{z_0} \frac{R_0}{R_1} \, dz_0$$

Any suffix other than zero refers to the value of this parameter at some particular level.

H : depth of fluid

$\vec{\omega}$: vorticity

$D/Dt = \mathbf{v}.\ \mathbf{grad} + \partial/\partial t$

$\nabla : \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)$

$\nabla^2 : \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$

$$(\nabla z_0)^2 = \left[\left(\frac{\partial z_0}{\partial x}\right)^2 + \left(\frac{\partial z_0}{\partial z}\right)^2 \right]$$

2. Introduction and survey of the general problem

The equations of hydrodynamics, although highly complex, do nevertheless provide for simple patterns of flow as stationary solutions to certain chosen problems. These patterns greatly depend on the various parameters governing them, but outside certain ranges, these cannot be realised. It is not the subject of this paper to consider the nature of instability of a fluid, although it may be remarked that, in general, a fluid does reflect unstable characteristics.

Probably the finest example of a fluid portraying stable characteristics in flows between parallel planes, is to be encountered when dealing with problems on stratification.

Illustrations of density variations having influence on fluid motion are provided by the passage past an obstacle of a fluid stream whose density varies continuously and stably.

The movement of stratified fluids with laminar motion has been studied by several research workers in the past. For example, Taylor, Goldstein and Prandtl, all made pioneering contributions in this field of study. The conception of Richardson's number helped in the understanding of the criterion of stability in that it demonstrated how stable stratification could be regarded as a mechanism that opposed turbulence. The movement of fluids in shear layers of uniform density is allied to this field, and when there is stratification of this nature, it is natural to regard the vorticity field as conservative with regard to the components $\vec{\omega} = (\xi, \eta, \zeta)$ (of which only one component survives in two dimensional flow).

Interesting fields of flow can be set up, depending on the nature of the stratification as well as that of the disturbance.

It has been remarked, for example by Taylor that it is a relatively simple matter to write out the equations which must be satisfied when we are dealing with such effects as variations in density and velocity of superposed, stratified (laminar) streams of fluid. But it is the nature of the solutions and their (realistic) interpretation that becomes a matter of considerable difficulty. Recourse to observation is much required to guide the search along channels of practical reality.

Taylor (1931) has remarked that 'the difficulty of solving the equations of motion of a fluid of variable density and velocity in any but the simplest cases leads one to consider problems of a more artificial nature which might perhaps be (more) completely soluble, while at the same time retaining some of the dynamical characteristics of the problem'.

Recourse to this approach was shown [for example by Scorer (1949)] to be far more useful if not entirely comprehensive. This line of thought was further vindicated by Long (1953).

When the disturbing influence creates large amplitude waves, there is greater necessity for adopting Taylor's line of thought, which in this instance precludes the necessity of working from the general non-linear equation.

The atmosphere is a stratified medium with major changes at the tropopause, and much research along these lines has been inspired by meteorologists.

The original researches on stratified media, although of considerable theoretical interest, had certain self-imposed limitations. Lyra (1943) and Queney (1947) produced theoretical solutions of wave patterns which died away with distance down and upstream of the barrier. The solutions were based on the linearised form of equations, derived from perturbation theory, which assumed that the amplitude of waves was small compared with the wavelength. The mathematics was made tractable by assuming the simplest possible physical conditions, in which L_0^2 was made constant independent of height, and with this assumption, the wave motion set up by a mountain was found to consist not of one single harmonic wave as in the case of a free surface, considered [as for example by Rayleigh and Kelvin (1883) etc] but a whole spectrum of harmonic waves, which interacted and cancelled each other some distance from the mountain. The results were of theoretical interest but certain inadequacies were at once apparent. Observation shows that in the troposphere, there is a wave motion, mainly composed of one harmonic wave (known as a lee wave), and is often of large amplitude, so as to make perturbation theory inadequate. Scorer (1949) in his theoretical study introduced a lee wave term of the type studied by Rayleigh and Kelvin, and although his fundamental equations were based on perturbation theory, he approached the general problem more realistically by considering a model of two layers instead of one, and in which L_0^2 was constant in each. But the main value of his contribution (Scorer 1949) was to demonstrate the importance of the lee wave term, showing that it is possible to have these waves extending in theory to infinity downstream.

The credit for developing a theory for waves of large amplitude goes to Long (1953). In his paper dealing with the flow of incompressible fluids, the exact, steady-state equations of motion and continuity of a perfect fluid moving two-dimensionally, with an arbitrary vertical distribution of density and velocity, were integrated once to yield a second order non-linear partial differential wave equation. The equation was examined with regard to uniqueness and stability of motion, and also this equation was used for deriving a sufficient condition that a given solution for motion in a channel is a unique steady solution. A perturbation approach for examining the stability of superposed streams of fluid had already been made by Taylor (1931) who also derived a general three-dimensional perturbation wave equation for stratified fluids.

The work of Long and Yih has been discussed in a later section, but we may remark here that together with the earlier work of Scorer, a definite breakthrough was made in the understanding of the theory of waves of large amplitude.

3. Air pocket simulation

We should make a remark about the rotor phenomenon. As stated previously, nearly all early contributions on this subject were based on the method of small perturbations and linearised differential equations. The basic assumptions are in contrast to the theory of waves of large amplitude. But when attempting to explain the phenomenon of air pockets did theoreticians start adopting the line of thought that further investigation was needed.

An air pocket is a region in which there is reversed flow. In his theoretical study, Lyra (1943) calculated the pressure disturbance at the surface in the lee, and regarded the rotor

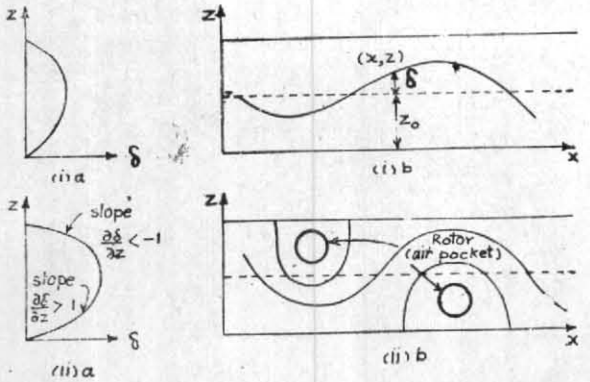

Fig. 1

Diagram (i) b shows a typical lee wave streamline. The wave is stable for small amplitudes if the stratification parameters of the fluid exceeds a certain value, which in the simplest case is given by :

$$G_e > \pi^2/H^2$$

When large amplitude waves are set up (as in diagram (ii) b), there are regions in the fluid where

$$\partial\delta/\partial z > \text{or} < \pm 1.$$

In this case rotors or air pockets are set up in the appropriate regions. These air pockets are strong mixing regions in a stably stratified fluid.

The dotted lines in the diagram represent streamlines before wave displacement has occurred.

as a flow consequent upon separation within a viscous boundary layer (Fig. 1).

The pressure distribution is linked with the vertical rate of change of streamline displacement $\partial\delta/\partial z$ which one recognises intuitively as a highly relevant quantity in the formation of rotors. We see that the larger $\partial\delta/\partial z$ is, the larger will be the pressure disturbance p, and accordingly the greater the likelihood of reversed flow near the surface and the formation of air pockets. Queney (1955) examined the development of a system of stationary waves and vortices in a basic flow having zero velocity at a particular level. The phenomenon of a pocket developing under highly stable conditions in inviscid fluids was finally explained by Long, who found that the stratified fluid when developing waves of sufficient amplitude, could be set in rotory motion.

Scorer (1958) examining the phenomenon of rotors from the point of wave amplitude, arrived at the following conditions —

$$\partial\delta/\partial z > \text{or} < \pm 1$$

according as $\delta >$ or < 0.

Where, $\partial\delta/\partial z = \pm 1$,

the flow is vertical, and this criterion is useful when examining theoretical solutions.

From practical considerations, it should be noted that the air pocket is a highly turbulent region with great instability and that theoretical solutions should be regarded as a guide to the positioning and formation only.

4. Researches on the wave equation

(a) Derivation of equation for the steady state

In this paper, we shall be concerned with a problem of fluid mechanics, in which the theoretical aspects shall follow as far as possible a study of ideal fluids, macroscopic and in continuum.

As in other branches in this subject, we start by assuming the general equations of motion of Navier-Stokes as the basis of theoretical considerations.

The velocity vector **v** then satisfied :

$$D\mathbf{v}/Dt = \partial\mathbf{v}/\partial t + (\mathbf{v}.\ \mathbf{grad})\,\mathbf{v} = -\frac{1}{\rho}\ \mathbf{grad}\,p + \mathbf{g} + \nu\nabla^2\mathbf{v} \qquad (4a\text{-}1)$$

The first two terms on the right-hand side express the rate of variation of **v** in consequence of the external forces (in our case the gravity field), and of the instantaneous pressure distribution. The term $\nu\nabla^2\,\mathbf{v}$ gives an additional variation due to viscosity, and is analogous to variations due to temperature in thermal conduction or to density in diffusion.

If we ignore viscous effects, then we may arrive at the vorticity equation for an incompressible fluid:

$$D\vec{\omega}/Dt = (\vec{\omega}.\ \mathbf{grad})\,\mathbf{v} + \mathbf{grad}\,(\log\rho)\times[\mathbf{g}-(D\mathbf{v}/Dt)] \qquad (4a\text{-}2)$$

Only one component survives in two-dimensional flow :

$$(\mathbf{v}.\ \mathbf{grad})\ \omega_y = [\mathbf{grad}\,(\log\rho)\times[\mathbf{g}-(D\mathbf{v}/Dt)]_y \qquad (4a\text{-}3)$$

By considering the continuity along a stream-tube, such that

$$u\,dz = U_o\,dz_o \qquad (4a\text{-}4)$$

we may write the surviving vorticity component in the form :

$$\begin{aligned}\omega_y &= (\partial u/\partial z) - (\partial w/\partial x)\\ &= U\nabla_1^2 z_o - U/(\nabla_1 z_o)^2\end{aligned} \qquad (4a\text{-}5)$$

Writing

$$\text{grad}\log\rho = -\beta_o\ \text{grad}\,z_o \qquad (4a\text{-}6)$$

where $\beta_o = -(\partial/\partial z_o)\log\rho_o$ is the static stability, the vorticity equation may then be written afte a few steps of algebra as :

$$0 = -\mathbf{v}_1 \text{ grad } \left\{ \omega_y - \frac{\beta_0}{2U} v_1^2 - \frac{g\,\beta_0}{U} z \right\} \tag{4a-7}$$

Hence, $\omega_y - \frac{\beta_0}{2U} v_1^2 - \frac{g\,\beta_0}{U} z =$ constant

along a streamline in the x—z plane. (4a-8)

Noting that $\mathbf{v}_1^2 = U_0^2 (\nabla_1 z_0)^2$ (4a-9)

after dropping the suffix, and omitting the vector notation, we have :

$$\nabla^2 z_0 + \left(\frac{U_0'}{U_0} - \frac{\beta_0}{z} \right) (\nabla z_0)^2 - \frac{g\beta_0}{U_0^2} z$$
$$= \text{constant along a streamline.} \tag{4a-10}$$

Defining two Lagrangian quantities G_0 and R_0 that are constant along a streamline, Eq. (4a-10) can be written as

$$\nabla^2 z_0 + \frac{R_0'}{R_0} (\nabla z_0)^2 - G_0 z$$
$$= \text{constant along a streamline} \tag{4a-11}$$

For upstream, if the streamlines are horizontal and undisturbed, then Eq. (4a-11) is given by:

$$\nabla^2 z_0 + \frac{R_0'}{R_0} (\nabla z_0)^2 - G_0 z = R_0'/R_0 - G_0 z_0 \tag{4a-12}$$

This is the required wave equation, and is also often written in an alternative form :

$$\nabla^2 \delta - \frac{R_0'}{R_0} \left\{ (\nabla \delta)^2 - 2 \frac{\delta}{\partial z} \right\} + G_0 \delta = 0 \tag{4a-13}$$

Equation (4a-13) follows from the fact that

$$\delta = z - z_0 \tag{4a-14}$$

The wave equation was first derived by Long (1953).

(b) *Analysis in terms of simple modes*

It was discovered by Long that the general wave equation could be rendered linear for a very important, but special, case where we could write it in the form :

$$\nabla^2 \delta + G_0 \delta = 0 \tag{4b-1}$$

where G_0 would be constant throughout the fluid. Generalised solutions have invariably produced most unsatisfactory results near the disturbing influence itself. But the natural lee wave terms may be drown out for investigation.

For plane parallel flows, in a fluid of depth H, it may be shown that :

$$\sin \nu H = 0 \tag{4b-2}$$

where,

$$\nu = \sqrt{G_0 - k^2} \tag{4b-3}$$

The wave number k is thus given by

$$k^2 = G_0 - (n^2\pi^2/H^2), \quad n = 1, 2, \ldots \tag{4b-4}$$

Thus for waves of this nature to occur, we must have $G_0 > k^2$ throughout. Such a model can be readily investigated by experiment.

Anaysis of this nature indicates that we may consider waves of the form :

$$\delta = \delta(z) \cos kx \tag{4b-5}$$

When we have systems that are non-linear, relationships involving the variables become far more complex. A non-linear system does not allow for superposition of solutions, and each problem has, therefore, to be treated on its own. There are, moreover, so many types of non-linearity, that to give general methods for attempting to solve them is very difficult, and nor is it possible to make any general remark concerning the form and nature of the solution even if available.

It is possible in this instance to consider certain simple cases of non-linearity by means of relatively unsophisticated techniques.

It may be suggested that any periodic function can be expressed as a Fourier series, and if the the non-linearity is small, the 'fundamental' would tend to dominate the series. A process of successive approximations could be expected to converge rapidly.

We would thus consider a series of the form :

$$\delta = \delta_0(z) + \delta_1(z) \cos kx + \delta_2(z) \cos 2kx + \delta_3(z) \cos 3kx + \ldots\ldots \tag{4b-6}$$

which would satisfy the wave equation, and throw out the unimportant terms, rather in the same manner as in perturbation techniques.

In problems such as those under study here, it may be possible to choose initial values near to the linearised case, and then proceed to an approximate solution.

(c) *Exact solutions*

We have Long's wave equation in the form:

$$\nabla^2 z_0 + \frac{R_0'}{R_0}\left(\nabla z_0\right)^2 - G_0 z = \frac{R_0'}{R_0} - G_0 z_0 \quad \text{(4c-1)}$$

We wish to transform equation (4c-1) to a new variable $\eta_0 = \eta_0(z_0)$ such that the second degree terms such as $(\nabla z_0)^2$ are absent in the new equation. This would greatly simplify the non-linear equation.

If $$\eta_0 = \eta_0(z_0) \quad \text{(4c-2)}$$

then

$$\eta_0 = \int_{z_1}^{z_0} \eta_0' \, dz_0 = \int_{z_1}^{z_0} f_0 \, dz_0 \quad \text{(4c-3)}$$

where $f_0(z_0)$ is to be found. From Eq. (4c-3),

$$\nabla \eta_0 = f_0 \nabla z_0 \quad \text{(4c-4)}$$

$$\begin{aligned} \therefore \nabla^2 \eta_0 &= \nabla \cdot (\nabla \eta_0) \\ &= \nabla \cdot (f_0 \nabla z_0) \\ &= f_0 \nabla^2 z_0 + f_0' (\nabla z_0)^2 \end{aligned} \quad \text{(4c-5)}$$

$$\text{or } \nabla^2 z_0 = \left(\frac{1}{f_0}\right) \nabla^2 \eta_0 - \left(\frac{f_0'}{f_0}\right)(\nabla z_0)^2 \quad \text{(4c-6)}$$

Equation (4c-1) may be re-written:

$$\left(\frac{1}{f_0}\right) \nabla^2 \eta_0 - \left(\frac{f_0'}{f_0}\right)(\nabla z_0)^2 + \left(\frac{R_0'}{R_0}\right)(\nabla z_0)^2 - \frac{R_0'}{R_0} + G_0(z_0 - z) = 0 \quad \text{(4c-7)}$$

For the terms in $(\nabla z_0)^2$ to vanish from Eq. (4c-7) we must have:

$$f_0'/f_0 = R_0'/R_0 \quad \text{(4c-8)}$$

$$\text{i.e.,} \quad f_0 = cR_0 \quad \text{(4c-9)}$$

where c is a constant.

Hence the only transformation through which $(\nabla z_0)^2$ can be made to vanish from Eq. (4c-1) is given by,

$$\eta_0 = \int_{z_1}^{z_0} f_0 \, dz_0 \quad \text{(4c-10)}$$

where $f_0 = cR_0$.

z_1 is a reference level, chosen as datum. Without loss of generality, we make $c = R_1^{-1}$ (the value of R_0^{-1} at $z_0 = z_1$) and write the above transformation as:

$$\eta_0 = \int_{z_1}^{z_0} \frac{R_0}{R_1} \, dz_0 \quad \text{(4c-11)}$$

The wave equation (4c-1) then becomes:

$$\nabla^2 \eta_0 - \frac{G_0 R_0 z}{R_1} + \frac{G_0 R_0 z_0}{R_1} - \frac{R_0'}{R_1} = 0 \quad \text{(4c-12)}$$

which is linear in η_0 if $G_0 R_0$ and $(G_0 R_0 z_0 - R_0')$ are linear in η_0. To satisfy these requirements, the following two conditions are specified:

$$G_0 R_0 = (L^{-2} m^3 \eta_0 + 1) G_1 R_1 \quad \text{(4c-13)}$$

$$G_0 R_0 z_0 - R_0' = (L^{-2} n^2 \eta_0 + z_1) G_1 R_1 - R_1' \quad \text{(4c-14)}$$

where $L^2 = G_1$ and L^2, m^2, n^2 which have dimensions $[L^{-2}]$, are appropriate physical constants, whose numerical values can be assumed in order to obtain the desired stream profiles. Eq. (4c-1) is linear in z_0 only when R_0 is constant and $m^3 = 0$. Whereas conditions (4c-13) and (4c-14) provide a much wider class of stream profiles by which equation (4c-12) can be made linear in η_0.

To satisfy equations (4c-13) and (4c-14), we must choose appropriate stream profiles.

From Eq. (4c-13), we have:

$$(G_0 R_0)' = m^3 R_0 \quad \text{(4c-15)}$$

and

$$(G_0 R_0)'' = m^3 R_0' \quad \text{(4c-16)}$$

If we now differentiate equation (4c-14) twice with respect to z_0, we have:

$$(G_0 R_0)' z_0 + G_0 R_0 - R_0'' = n^2 R_0 \quad \text{(4c-17)}$$

and

$$(G_0 R_0)'' z_0 + z(G_0 R_0)' - R_0''' = n^2 R_0' \quad \text{(4c-18)}$$

From equations (4c-15), (4c-16) and (4c-18) we derive an equation for R_0 independent of G_0:

$$R_0''' + (n^2 - m^3 z_0) R_0' - z\, m^3 R_0 = 0 \quad \text{(4c-19)}$$

A suitable choice of n^2 and m^3 for a realistic physical case with triple boundary conditions of R_0 are required for the solution of equation (4c-19) which can be obtained numerically. From equation (4c-11), we have a relation for $\eta_0 - z_0$.

An equation for η_0-z can be obtained from the equations (4c-12), (4c-13) and (4c-14). Re-writing (4c-12), we have :

$$\nabla^2\eta_0 - G_1 (L^{-2}m^3\eta_0 + 1) z + G_1 (z_1 + L^{-2}n^2\eta_0) - R_1'/R_1 = 0 \qquad (4c\text{-}20)$$

Putting $z_1=0$, Eq. (4c-20) can be written,

$$\nabla^2\eta_0 + (n^2 - m^3 z)\,\eta_0 = G_0 z + R_1'/R_1 \qquad (4c\text{-}21)$$

This equation is not homogeneous, and so we cannot write $\eta_0 \propto \cos kx$. Instead, we write :

$$\eta_0 = A(z) + w(x, z) \qquad (4c\text{-}22)$$

where w is a solution of the homogeneous equation,

$$(\nabla^2 + n^2 - m^3 z)\,\eta_0 = 0 \qquad (4c\text{-}23)$$

and A is a particular integral. Then $w \propto \cos kx$, and

$$A'' + (n^2 - m^3 z)\,A = G_1 z_0 + R_1'/R_1 \qquad (4c\text{-}24)$$

$$w'' + (n^2 - m^3 z - k^2) w = 0 \qquad (4c\text{-}25)$$

with suitable boundary conditions. The function w represents the lee waves, and A is the stream function of the undisturbed flow, that is $\eta_0(z_0)$. This can be seen from equations (4c-13) and (4c-14) by simply eliminating G_0R_0 from them and using (4c-11); this leads to

$$\frac{d^2\eta_0}{dz_0^2} + (n^2 - m^3 z_0)\eta_0 = G_1 z_0 + R_1'/R_1 \qquad (4c\text{-}26)$$

We thus have a relation between η_0 and z_0, and also between η_0 and z. Hence we have a relation between z and z_0, and so streamlines can be drawn for a particular flow.

Solutions to equations (4c-24), (4c-25) and (4c-26) involve Airy and associated functions, or less elegantly Bessel functions of order 1/3.

Considering Eq. (4c-24), we have :

$$A = a_A\, A_i(Z_A) + b_A\, B_i(Z_A) + C_A\, \pi\, G_i(Z_A) - d_A \qquad (4c\text{-}27)$$

where,

$$Z_A = m\left(z - \frac{n^2}{m^3}\right) \qquad (4c\text{-}28)$$

$$C_A = -\frac{R_1'}{m^2 R_1} + \frac{G_1 n^2}{m^3} \qquad (4c\text{-}29)$$

$$d_A = G_1/m^3 \qquad (4c\text{-}30)$$

and a_A, b_A are constants depending on the boundary conditions. The integral G_i is defined in real form as

$$G_i(Z_A) = \frac{1}{\pi}\int_0^\infty \sin\left(Z_A t + \frac{t^3}{3}\right) dt \qquad (4c\text{-}31)$$

and is the complex part of an expression of which $A_i(Z_A)$ is the real part. Scorer (1950) has tabulated this function over a suitable range. In the same paper, defining the function $H_i(Z_A)$ as,

$$H_i(Z_A) = B_i(Z_A) - G_i(Z_A) \qquad (4c\text{-}32)$$

Scorer tabulated equation (4c-32) for ranges $Z_A > 0$ rather than $G_i(Z_A)$ because $G_i(Z_A)$ diverges like $B_i(Z_A)$, while $H_i(Z_A)$ decreases monotonically as Z_A increases.

Considering Eq. (4c-25), we have,

$$w = a_w\, A_i(Z_w) + b_w\, B_i(Z_w) \qquad (4c\text{-}33)$$

where,

$$Z_w = m\left(z - \frac{n^2 - k^2}{m^3}\right) \qquad (4c\text{-}34)$$

and a_w, b_w are constants determined by the boundary conditions, and k is the wave number.

Eq. (4c-26) has a solution as follows:

$$\eta_0 = a\eta_0\, A_i(Z_0) + b\eta_0\, B_i(Z_0) + c\eta_0\, \pi\, G_t(Z_0) - d\eta_0 \qquad (4c\text{-}35)$$

where $a\eta_0$, $b\eta_0$ are arbitrary constants depending on the boundary conditions, and

$$Z_0 = m\left(z_0 - \frac{n^2}{m^3}\right) \qquad (4c\text{-}36)$$

$$c\eta_0 = -\frac{R_1'}{m^2 R_1} + \frac{G_1 n^2}{m^5} \qquad (4c\text{-}37)$$

$$d\eta_0 = G_1/m^3 \qquad (4c\text{-}38)$$

The boundary conditions for the problem under study are

$$\eta_0(z_0 = 0) = 0 \qquad (4c\text{-}39)$$

$$\eta_0(z_0 = H) = \eta_0(z = H) \qquad (4c\text{-}40)$$

from which a final solution for η_0 may be obtained.

Reference should be made to the work of Yih (1960) who made use of modified Bessel functions for a solution to a similar problem.

5. Conclusions

The work in this paper has been based primarily on the researches carried out by Scorer, Long and Yih.

In Yih's papers, much generalising was adopted and it may be remarked that from a formal point of view, the linearised problem was adequately tied up. But the nature of these waves could be investigated further by actual calculations, and there exists much scope for such work, preferably with the aid of an electronic computer.

It may also be remarked that Long generalised the entire problem in his 1953 paper, and certainly his 1958 note to the *Quarterly Journal of the Royal Meteorological Society* contained most of the material which Yih in his later work developed independently.

The non-linear equation deserves to be investigated in conjunction with a suitable apparatus model that may be available for generating such waves.

It should, however, be remarked that Long's original discovery of the special case, whereby linear solutions could be investigated, was a very important discovery in this research, and fulfils the original remarks of Taylor (1931) mentioned in the introduction. In fact, a remark made by Scorer in his 1949 paper to the effect that a more general equation would not serve a necessarily better purpose, carried much deep foresight into this problem of waves in stratified media.

Acknowledgements

The author would like to express his gratitude to Professor R. S. Scorer of the Imperial College for much inspiration expressed in this paper. The author is also beholden to Professor P. A. Shepperd, F. R. S., in whose department he held the post of Research Assistant some years ago.

The work was completed in the Department of Mechanical Engineering and Naval Architecture, Portsmouth Polytechnic, and the author would like to express his gratitude to Dr. D. C. Chandler and also to his predecessor, the late Mr. E. D. Rees.

REFERENCES

Long, R. R.	1953	*Tellus*, **5**, pp. 42-58.
	1958	*Quart. J. R. met. Soc.*, **84**, pp. 159-161.
Lyra, G.	1943	*See* Scorer (1949)
Queney	1953	
Rayleigh, and Kelvin	1883	
Scorer, R. S.	1949	*Quart. J. R. met. Soc.*, **75**, pp. 41-56.
Taylor, G. I.	1931	*Proc. Roy. Soc. London*, (**A**) **132**, p. 499.
Yih, C. S.	1960	*J. Fluid Mech.*, **9**, pp. 161-174.