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Letters to the Editor

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THE VORTICITY EQUATION

1. In this note we show that the full vorticity equation for horizontal motion can be derived in a simple way starting from the concept of circulation round the periphery of an elementary area. As we have not come across such a derivation in the usual text books we feel that this might be of interest to readers, especially because the physical meaning of the different terms is clearly brought out. This may be regarded as an extension of the treatment given by Petterssen (1956).

2. Consider the fluid particles comprising an elementary vector area $\mathbf{A}$ around a point P in the atmosphere at latitude ϕ and longitude λ. If $\mathbf{Q}$ is the absolute vorticity (three-dimensional) at P, then the absolute circulation C_a around the periphery of the area is given by :

$$C_a = \mathbf{A} \cdot \mathbf{Q} \tag{1}$$

where,

$$\mathbf{Q} = \nabla \times \mathbf{V} + 2\,\vec{\Omega} \tag{2}$$

Here $\mathbf{V}$ is the velocity (three-dimensional) at P and $\vec{\Omega}$ is the angular velocity of rotation of the earth. If u, v, w are the components of $\mathbf{V}$ in a spherical co-ordinate system with axes towards the east, north and vertical respectively, and if the radius of the earth is a, then :

$$\begin{aligned}\mathbf{Q} &= \mathbf{i}\,Q_x + \mathbf{j}\,Q_y + \mathbf{k}\,Q_z \\ &= \mathbf{i}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \frac{v}{a}\right] + \mathbf{j}\left[\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{u}{a} + 2\,\Omega\cos\phi\right] + \mathbf{k}\left[\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{u\tan\phi}{a} + 2\Omega\sin\phi\right]\end{aligned} \tag{3}$$

where $dx = a\cos\phi\, d\lambda$; $dy = a\, d\phi$

Differentiating (1) with respect to time :

$$\frac{dC_a}{dt} = \frac{d\mathbf{A}}{dt}\cdot\mathbf{Q} + \mathbf{A}\cdot\frac{d\mathbf{Q}}{dt} = \frac{dA}{dt}\,\mathbf{n}\cdot\mathbf{Q} + A\frac{d\mathbf{n}}{dt}\cdot\mathbf{Q} + \mathbf{A}\cdot\frac{d\mathbf{Q}}{dt} \tag{4}$$

where $\mathbf{n}$ is unit vector normal to the area A.

3. We shall now specify that the elementary area $\mathbf{A}$ is horizontal so that $\mathbf{n} = \mathbf{k}$. Then :

$$\frac{dA}{dt}\,\mathbf{n}\,.\,\mathbf{Q} = \frac{dA}{dt}\,Q_z \tag{5}$$

$d\mathbf{n}/dt$ results from the tilt of the area A about the vertical; this vector is perpendicular to $\mathbf{n}$. If δx and δy are the edges of the element A, then a differential vertical velocity δw between the ends of δx or δy will tilt the vector $\mathbf{n}$ in the x or y directions. It can be readily seen that :

$$\frac{d\mathbf{n}}{dt} = -\,\mathbf{i}\,\frac{\partial w}{\partial x} - \mathbf{j}\,\frac{\partial w}{\partial y} \tag{6}$$

Hence

$$A\,\frac{d\mathbf{n}}{dt}\cdot\mathbf{Q} = -A\left[\frac{\partial w}{\partial x}\,Q_x + \frac{\partial w}{\partial y}\,Q_y\right] \tag{7}$$

Finally

$$\begin{aligned}\mathbf{A}\,.\,\frac{d\mathbf{Q}}{dt} &= \mathbf{A}\cdot\frac{d}{dt}\left[\,\mathbf{i}\,Q_x + \mathbf{j}\,Q_y + \mathbf{k}\,Q_z\right] \\ &= A\,\mathbf{k}\cdot\left[\,\mathbf{i}\,\frac{dQ_x}{dt} + \mathbf{j}\,\frac{dQ_y}{dt} + \mathbf{k}\,\frac{dQ_z}{dt} + Q_x\,\frac{d\mathbf{i}}{dt} + Q_y\,\frac{d\mathbf{j}}{dt} + Q_z\,\frac{d\mathbf{k}}{dt}\right] \\ &= A\,\frac{dQ_z}{dt} + A\,\mathbf{k}\,.\left[\,Q_x\frac{d\,\mathbf{i}}{dt} + Q_y\frac{d\,\mathbf{j}}{dt}\right]\end{aligned} \tag{8}$$

For motion on the spherical earth of radius a it can be readily shown that (Haltiner and Martin 1957) :

$$\left.\begin{aligned}\frac{d\,\mathbf{i}}{dt} &= \frac{u}{a\cos\phi}\,(\sin\phi\,\mathbf{j} - \cos\phi\,\mathbf{k}) \\ \frac{d\,\mathbf{j}}{dt} &= -\,\frac{u\tan\phi}{a}\,\mathbf{i} - \frac{v}{a}\,\mathbf{k}\end{aligned}\right\} \tag{9}$$

Substituting (9) in (8) and simplifying we ge :

$$\mathbf{A}\cdot\frac{d\mathbf{Q}}{dt} = A\,\frac{dQ_z}{dt} - A\left[\,\frac{u}{a}Q_x + \frac{v}{a}\,Q_y\,\right] \tag{10}$$

We thus have :

$$\frac{dC_a}{dt} = \frac{dA}{dt} Q_z + A \frac{dQ_z}{dt} - A\left[\left(\frac{\partial w}{\partial x} + \frac{u}{a}\right) Q_x + \left(\frac{\partial w}{\partial y} + \frac{v}{a}\right) Q_y \right] \quad (11)$$

4. According to Bjerknes' Circulation Theorem, dC_a/dt is the number of (α, p) solenoids (S) intersected by the area A. Hence we have :

$$\frac{dQ_z}{dt} = -\frac{1}{A}\frac{dA}{dt} Q_z + \frac{S}{A} + \left(\frac{\partial w}{\partial x} + \frac{u}{a}\right) Q_x + \left(\frac{\partial w}{\partial y} + \frac{v}{a}\right) Q_y \quad (12)$$

S/A is the number of solenoids intersected by unit horizontal area or the vertical component N_z of the solenoidal vector; its magnitude is :

$$-\left(\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}\right) \quad \text{and} \quad \frac{1}{A}\frac{dA}{dt}$$

has the meaning of the horizontal divergence $\nabla_H \cdot \mathbf{V}$ when the area is infinitesimal. If $Q_z = \zeta + f$,

where $\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{u \tan\phi}{a}$ and $f = 2\,\Omega \sin\phi$

we have :

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\nabla_H \cdot \mathbf{V} + \left(\frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y}\right) + \left[\left(\frac{\partial w}{\partial x} + \frac{u}{a}\right) Q_x + \left(\frac{\partial w}{\partial y} + \frac{v}{a}\right) Q_y \right] \quad (13)$$

This is the full vorticity equation giving the rate of change of absolute vorticity (vertical component) of an individual fluid element in motion.

5. The three terms on the right hand side are respectively the divergence, the solenoidal and tilting terms. The tilting term can be expanded by substituting the values of Q_x and Q_y given in Eq. (3). We find that :

$$\left(\frac{\partial w}{\partial x} + \frac{u}{a}\right) Q_x + \left(\frac{\partial w}{\partial y} + \frac{v}{a}\right) Q_y = \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\frac{u}{a} - \frac{\partial w}{\partial x}\frac{v}{a}\right) + 2\,\Omega\cos\phi\,\frac{\partial w}{\partial y} + \left(\frac{u}{a} Q_x + \frac{v}{a} Q_y\right) \quad (14)$$

Of the four terms on the right hand side, the first term is main tilting term that is given in the usual derivations of the vorticity equation starting from the equations of motion in simplified form (Haltiner and Martin 1957). The second term which has a form similar to the first, results from the sphericity of the earth. The term $2\,\Omega\cos\phi\,(\partial w/\partial y)$ is the change in the contribution to the vorticity of the element from the earth's rotation due to the tilting of the element in the y-direction. The terms $(u/a)\,Q_x$ and $(v/a)\,Q_y$ arise from the motion of the element in the x and y directions which alters the direction of its normal on the spherical earth and so gives rise to contributions to Q_z from Q_x and Q_y. The three additional tilting terms that occur in the full vorticity equation as we have derived above are, of course, much smaller than the main tilting term.

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