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An optimum estimate of the periodicity of simple vibrating systems

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ABSTRACT. In this paper, we present a method of estimating the characteristics of simple oscillating systems by the principle of least squares. It is shown that the mean of two successive positions of an oscillating particle does not always provide the best estimate. Weighting factors for computing the best estimate have been computed for (*i*) a system oscillating with unit frequency, (*ii*) a system oscillating with any frequency and (*iii*) Rossby waves in the atmosphere.

1. Introduction

In meteorological dynamics we often encounter the problem of finding an optimum estimate of equations which specify the propagation of atmospheric waves. In this report, we propose a scheme for computing optimum estimates based on the principle of least squares. More specifically, we pose the following problem.

2. Vibrating system with unit frequency

Let us consider a simple vibrating system governed by the equation—

$$d^2x/dt^2 + x = 0 \tag{2·1}$$

The position of a particle (x) and its velocity (u) are then defined by the following functions of time (t)—

$$x = A\cos t + B\sin t \tag{2·2a}$$

$$u = -A\sin t + B\cos t \tag{2·2b}$$

where, A and B represent constants.

Let us suppose we do not know A or B, nor x and u; but we do have the following information—

(*i*) at $t = 0$, $x = x_0 - \delta_0$ (2·3 a)

$u = u_0 - \epsilon_0$ (2·3 b)

(*ii*) at $t = t_1$, $x = x_1 - \delta_1$ (2·4 a)

$u = u_1 - \epsilon_1$ (2.4 b)

where x_0, x_1 and u_0, u_1 represent observed positions and velocities, and δ_0, δ_1, ϵ_0, ϵ_1 are 'errors' in observations. The problem, which we propose to discuss, is to decide (*i*) what is the best estimated value of the constants A and B from the information available in (2·3 a, b) and (2·4 a, b), and (*ii*) what is the error in our estimate of $(A^2 + B^2)$.

Let us choose as our estimates the constants A' and B'. We then minimize the function,

$$f(A', B') = [x(t=0) - x_0]^2 + [x(t=t_1) - x_1]^2 + [u(t=0) - u_0]^2 + [u(t=t_1) - u_1]^2 \tag{2·5}$$

Noting that the necessary condition for a minima in $f(A', B')$ is

$$\delta f = \frac{\partial f}{\partial A'}\,\partial A' + \frac{\partial f}{\partial B'}\,\delta B' = 0$$

we have,

$$\partial f/\partial A' = \partial f/\delta B' = 0 \tag{2·6}$$

For a system vibrating with unit frequency, as in (2·1), we find

$$f(A', B') = (x_0^2 + x_1^2 + u_0^2 + u_1^2) + 2[(A')^2 - A'(x_0 + x_1\cos t_1 - u_1\sin t_1)] + 2[(B')^2 - B'(u_0 + x_1\sin t_1 + u_1\cos t_1)] \tag{2·7}$$

On evaluating the partial derivatives of (2·7) and applying (2·6), we find

$$A' = \tfrac{1}{2}(A_0 + A_1) \tag{2.8 a}$$

$$B' = \tfrac{1}{2}(B_0 + B_1) \tag{2·8 b}$$

where A_0, A_1, B_0, B_1 are values of the constants in (2·2 a,b) at $t=0$ and t_1 respectively. Thus, we infer that the best estimate for A and B is the average of the values recorded at $t=0$ and t_1.

We also note that the total error is—

$$E = \delta_0^2 + \delta_1^2 + \epsilon_0^2 + \epsilon_1^2 = f(A', B') \tag{2·9}$$

On using (2·8 a,b) as our estimate of A', B', we find that the total error is —

$$E = (x_0^2 + x_1^2 + u_0^2 + u_1^2) - \tfrac{1}{2}[(A_0 + A_1)^2 + (B_0 + B_1)^2] \tag{2·10}$$

TABLE 1

Best's estimates of A', B'

$\omega \triangle \dot{t}$	$\pi/4$	$\pi/2$	π
A'	$\frac{1}{1+3\omega^2}\left[2A_0 + A_1(1+\omega^2) + B_1(1-\omega^2)\right] + \frac{\omega^2(1-\omega^2)}{(3+\omega^2)(1+3\omega^2)}\left[2B_0 + B_1(1+\omega^2) + A_1(1-\omega^2)\right]$	$\frac{A_0 + A_1\omega^2}{1+\omega^2}$	$\frac{1}{2}(A_0 + A_1)$
B'	$\frac{\omega^2}{1+3\omega^2}\left[2B_0 + B_1(1+\omega^2) + A_1(1-\omega^2)\right]$	$\frac{1}{1+\omega^2}(B_0\omega^2 + B_1)$	$\frac{1}{2}(B_0 + B_1)$

TABLE 2

Best's estimates of A', B' (Rossby waves)

$(\chi = \alpha^2 c^2)$

$(\alpha c)\triangle t$	$\pi/4$	$\pi/2$	π
A'	$\left[A_0\chi(3+\chi) + 2A_1(1+3\chi) + B_0(\chi-1) + B_1(1-\chi)\right] \div (\chi+3)^2-8$	$\frac{\chi A_0 + A_1}{1+\chi}$	$\frac{1}{2}(A_0 + A_1)$
B'	$\left[A_0\chi(1-\chi) + 2A_1\chi(1-\chi) + B_0(1+3\chi) + 2B_1\chi(1+\chi)\right] \div (\chi+3)^2-8$	$\frac{B_0 + \chi B_1}{1+\chi}$	$\frac{1}{2}(B_0 + B_1)$

3. Vibrating system with a different frequency (ω)

The result indicated by (2·8 a, b) could have been deduced simply by physical intuition, because the best estimate is generally the average of observations recorded at two different times. But, we shall see presently, this simple result is not true when we consider systems that are not vibrating with unit frequency.

Let ω represent the frequency of the vibrating system, we have

$$d^2x/dt^2 + \omega^2 x = 0 \qquad (3\cdot1)$$

whence

$$x = A\cos\omega t + B\sin\omega t \qquad (3\cdot2\,a)$$

$$u = \omega(-A\sin\omega t + B\cos\omega t) \qquad (3.2b)$$

Proceeding as before, the function to be minimized is —

$$\begin{aligned} f(A', B') &= (x_0^2 + x_1^2 + u_0^2 + u_1^2) + (A')^2 \times \\ &\times(1+\cos^2\omega\triangle t + \omega^2\sin^2\omega\triangle t) - 2A'(x_0 + \\ &+ x_1\cos\omega\triangle t - \omega u_1\sin\omega\triangle t + B'\cos\omega\triangle t\times \\ &\times\sin\omega\triangle t + B'\omega^2\cos\omega\triangle t\sin\omega\triangle t) + \\ &+ (B')^2(\omega^2 + \sin^2\omega\triangle t + \omega^2\cos^2\omega\triangle t) - \\ &- 2B'(x_1\sin\omega\triangle t + \omega u_0 + \omega u_1\cos\omega\triangle t) \qquad (3\cdot3) \end{aligned}$$

In (3·3), $\triangle t$ denotes the time interval. Upon evaluating the partial derivatives of (3·3) and applying (2·6) we find —

$$A'(1+\cos^2\omega\triangle t + \omega^2\sin^2\omega\triangle t) - B'(1-\omega^2)\times$$
$$\sin\omega\triangle t\cos\omega\triangle t = x_0 + (x_1\cos\omega\triangle t - \omega u_1\sin\omega\triangle t)$$

$$B'(\omega^2+\sin^2\omega\triangle t+\omega^2\cos^2\omega\triangle t)$$
$$= \omega u_0 + (x_1\sin\omega\triangle t + \omega u_1\cos\omega\triangle t) \qquad (3\cdot4)$$

From (3·4) we note that if $\omega = 1$,

$$A' = \tfrac{1}{2}(A_0 + A_1),$$
$$B' = \tfrac{1}{2}(B_0 + B_1)$$

This is identical with our previous result (2·8 a,b). But, as we can readily see, if different values are assigned to $\omega\triangle t$, the best estimate of A', B' is not of necessity the average of A_0, A_1 or B_0, B_1. In Table 1, we show the best estimates of A', B' for three values of $\omega\triangle t$.

We see from Table 1 that the average of A_0, A_1 and B_0, B_1 is the best estimate for A', B' only when $\omega\triangle t=\pi$. At other times, we require different weighting factors to compute the best estimate.

4. Rossby waves

The foregoing analysis may be extended to Rossby waves in the atmosphere. We consider horizontal transverse waves travelling in the x direction, and assume that the u (eastward), v (northward) component of velocity is independent of the y component pointing north. Under these constraints, the conservation of absolute vorticity is expressed by —

$$\frac{\partial^2 v}{\partial x\,\partial t} + \bar{u}\,\frac{\partial^2 v}{\partial x^2} + \beta v = 0 \qquad (4\cdot1)$$

where $\beta\,(\partial f/\partial y)$ measures the variation of the Coriolis parameter (f) with latitude, and $\bar{u}$ is a constant zonal velocity. Equation (4·1) has periodic wave solutions of the form —

$$v = A\cos\alpha(x-ct) + B\sin\alpha(x-ct) \qquad (4\cdot2)$$

$$y = \frac{1}{\alpha c}[-A\sin\alpha(x-ct) + B\cos\alpha(x-ct)] \quad (4\cdot3)$$

By direct substitution of (4·2), (4·3) in (4·1) we observe the well known relation between wave velocity (c) and wave number (α),

$$c = \bar{u} - \beta/\alpha^2 \qquad (4\cdot4)$$

It is to be noted from (4·2) and (4·3) that when we study Rossby waves, the function to be minimized $f(A', B')$ is a function of both x and the frequency (α). We shall, therefore, obtain different sets of weighting factors for the best estimate of A', B' depending on the values assigned to αx. For simplicity, we have chosen $\alpha x=0$ in calculating the weighting factors. This reduces (4·2) and (4·3) to

$$v = A\cos\alpha ct - B\sin\alpha ct \qquad (4\cdot5)$$

$$y = \frac{1}{\alpha c}(A\sin\alpha ct - B\cos\alpha ct) \qquad (4\cdot6)$$

The above assumption does not involve any loss in generality, because the computations can be readily extended to other values of αx.

In Table 2 we show the best estimates of A', B' at $\alpha x=0$ for three different values of $\alpha c\triangle t$ where, as before, $\triangle t$ represents the time interval.

The interesting feature of Table 2 is that the average is again the best estimate when $\alpha c \triangle t = \pi$; at other times, different weighting factors are required to compute the best estimate. These weighting factors depend on the value of $\chi = \alpha^2 c^2$. For meteorological waves, we note that $|\chi| << 1$. If this assumption is introduced in Table 2, we see that the weighting factors (for $\alpha c \triangle t = \pi/4$) can be reduced to fairly simple algebraic expressions.

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